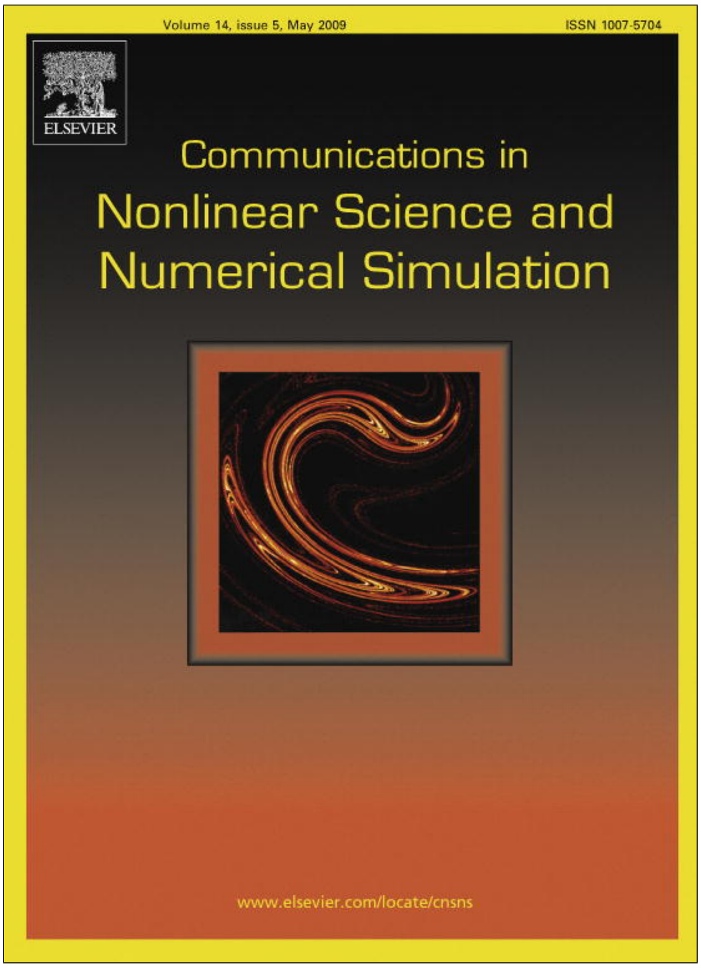
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# Quasi-periodic solutions and periodic bursters in quasiperiodically driven oscillators

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ABSTRACT

In this paper, we propose a perturbation method to determine an approximation and conditions of existence of quasi-periodic (QP) solutions and bursting dynamics in a quasi-periodically driven system. The QP forcing consists of two periodic excitations, one with a very slow frequency and the other with a frequency of the same order of the proper frequency of the oscillator. A first averaging is done over the fast dynamics, then the quasi-static solutions of the modulation equations of amplitude and phase are determined and their stability analyzed. We show that a necessary condition for the occurrence of periodic bursters is that the slow excitation is parametric.



## 1. Introduction

Quasi-periodically forced systems are those that are influenced by two periodic signals with incommensurate frequencies. A special interest has been given to the construction and the existence of regular motions through numerical and perturbation methods. Broer and Simó [1] geometrically explored resonance tongues containing instability pockets in a linear Hill's equation with quasi-periodic forcing. Rand et al. [2] determined an approximation of the regions of stability using Lyapunov exponents and the harmonic balance method. Belhaq and co-workers [3] approximated analytically QP solutions of a damped cubic nonlinear QP Mathieu equation, using the double perturbation method. This method uses two perturbation parameters to make natural the application of two reductions through perturbation methods [4]. This method was used when one of the two frequencies is of order $\mathcal{O}(\varepsilon)$.

In this paper, we study QP excitations consisting of a frequency $\nu$ of the same order of the proper frequency of the oscillator and a *very slow* frequency $\varepsilon^p$ where $p$ is an integer greater than 1. This very slow excitation induces quasi-static solutions on the slow manifold resulting from an averaging over the fast scale of time. Consequently, a change in the nature of the quasi-static solutions during a period of the very slow frequency, can lead to the appearance of periodic bursters. The resonant and non-resonant cases, as well as the external and parametric excitations, are discussed. Thus, we focus in this work on the conditions of existence and the construction of QP solutions and periodic bursters solutions. For detailed classification of bursters, see Golubitsky et al. [5].

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This paper is organized as follows: in Section 2 we state the main results related to the existence of QP solutions and periodic bursters using an average over the fast dynamic. In Section 3, two examples are given. The first is an oscillator with both parametric damping and nonlinear damping. The second example is a quasi-periodically excited van der Pol oscillator. In Section 4, a summary of the results is given.

## 2. Formulation of the method

We consider the following quasi-periodically driven system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \varepsilon\mathbf{g}(\nu t, \tau, \mathbf{x}; \mu), \tag{1}$$

where $\mathbf{x} \in \mathrm{IR}^2$, $\mathbf{f}(\mathbf{x})$ is a linear function of $\mathbf{x}$, $\tau = \varepsilon^p t$ with $p \geqslant 2$ and $\varepsilon$ is a small positive parameter, $\mu$ represents the set of parameters of the system and $\mathbf{g}$ is polynomial in $\mathbf{x}$ and $2\pi$-periodic in $\nu t$ and $\tau$. The frequency $\nu$ has the same order as the proper frequency of the system, that is assumed, to be equal to unity. We say that the function $\mathbf{g}$ is a QP function with basic frequencies $\nu$ and $\varepsilon^p$ i.e., it contains the terms $\cos(\nu t)$ and $\cos(\varepsilon^p t)$.

We assume that for $\varepsilon = 0$ Eq. (1) has an elliptic equilibrium $\mathbf{x} = \mathbf{0}$. In what follows, the multiple scales method (MSM) is used [6] to construct an approximation of solutions of Eq. (1). Thus, the solution $\mathbf{x}(t, \varepsilon)$ of Eq. (1) and the independent scales of time $T_i$ are expressed as follows:

$$\mathbf{x}(t; \varepsilon) = \sum_{i=0}^{N} \varepsilon^i \mathbf{x}_i(T_0, T_1, \ldots, T_N) + \mathcal{O}(\varepsilon^{N+1}), \tag{2}$$

$$T_i = \varepsilon^i t, \frac{\mathrm{d}}{\mathrm{d}t} = \sum_{i=0}^{N} \varepsilon^i D_i, \quad \text{where} \quad D_i = \frac{\partial}{\partial T_i} \tag{3}$$

The functions $\mathbf{x}_i$ are assumed to be periodic. Substituting Eqs. (2) and (3) into the original Eq. (1), we obtain the following hierarchy of problems:

$$\text{order } \mathcal{O}(1) : D_0\mathbf{x}_0 = \mathbf{f}(\mathbf{x}_0). \tag{4}$$

The unperturbed solution $\mathbf{x}_0$ is assumed, without loss of generality, to be periodic with the frequency 1.

$$\text{order } \mathcal{O}(\varepsilon) : D_0\mathbf{x}_1 = -D_1\mathbf{x}_0 + \mathbf{g}(\nu t, \tau, \mathbf{x}_0; \mu) + \frac{1}{\varepsilon}[\mathbf{f}(\mathbf{x}_0 + \varepsilon\mathbf{x}_1) - \mathbf{f}(\mathbf{x}_0)]. \tag{5}$$

The usual approach in studying such systems is based on the so-called quasi-steady state assumption which means that the fast variables are in a quasi-steady state; i.e., the fixed points are no more static points but they depend on the very slow excitation $\cos(\varepsilon^p t)$.

At this level of computations one should discuss the existence of resonances between the forcing $\cos(\nu t)$ and the unperturbed frequency.

### 2.1. Non-resonant case

The elimination condition of secular terms, from Eq. (5), can be written as

$$D_1\mathbf{x}_0 = \mathbf{g}^*(\tau, \mathbf{x}_0; \mu), \tag{6}$$

where the function $\mathbf{g}^*$ contains the contribution of odd terms with respect to $\mathbf{x}$ contained in the function $\mathbf{g}$. The number of the zeros of $\mathbf{g}^*(\tau, \mathbf{x}_0; \mu)$ determines the number of the solutions $\mathbf{x}$ to Eq. (1) that coexist for the same values of $\mu$.

Let $\mathbf{X}_0$ be a solution of $\mathbf{g}^*(\tau, \mathbf{X}_0; \mu) = 0$. The function $\mathbf{g}^*$ can depend on the slow time scale $\tau$ only when the slow excitation $\mathbf{g}$ contains a parametric term of the form $\cos(\varepsilon^p t)x^{2n+1}$ with $n \in \mathrm{IN}$. In this case, the solution $\mathbf{X}_0$ depends on the slow time scale and the set of parameters $\mu$ i.e., $\mathbf{X}_0(\tau; \mu)$. Hence, during a period of the slow time scale $2\pi/\varepsilon^p$, the solution $\mathbf{X}_0(\tau; \mu)$ can lose stability or disappear. When these processes lead to the stabilisation or the appearance of another bounded solution, the system (1) will have a periodic burster solution.

In the case where the slow excitation is external, the function $\mathbf{g}^*(\mathbf{x}_0; \mu)$ does not depend on the slow time scale $\tau$ and consequently its zeros $\mathbf{X}_0(\mu)$ also. Thus, the existence of periodic bursters is excluded in this case.

In the case of parametric slow excitation of the form $\cos(\varepsilon^p t)\mathbf{x}^{2n}$, the approximated solutions $\mathbf{x}$ Eq. (1) is three-period-QP with three fundamental frequencies 1, $\nu$ and $\varepsilon^p$.

In the case of parametric slow excitation of the form $\cos(\varepsilon^p t)\mathbf{x}^{2n+1}$, the approximated solution of $\mathbf{x}$ of Eq. (1) can be a periodic burster. Otherwise it is a phase slowly modulated solution.

### 2.2. Resonant case

Here we restrict our study to the case where $\nu = m + \varepsilon\sigma$ with $m \in \mathrm{IN}^*$ and $\sigma$ is a detuning parameter. The condition of elimination of secular terms can be written as follows:

$$D_1\mathbf{x}_0 = \mathbf{g}^*(\sigma T_1, \tau, \mathbf{x}_0; \mu). \tag{7}$$

The explicit dependence on the slow time scale $T_1$ in Eq. (7) can be eliminated through a change of variables.

- Case of external slow excitation

When the resonant excitation is external, the function $\mathbf{g}^*$ in the solvability condition (7) contains always the contribution of the odd terms in $\mathbf{x}_0$ and the contribution of the resonant forcing when $m = 1$. The approximated solution $\mathbf{x}$ of Eq. (1) is 2-period-QP with the two fundamental frequencies 1 and $\varepsilon^p$. It is 3-period-QP when $\mathbf{g}^*$ causes self-excited limit cycles.

When the resonant excitation is parametric i.e., of the form $\mathbf{x}_0^n \cos(\nu t)$ with $n$ an integer, it contributes to the function $\mathbf{g}^*$ in Eq. (7)in addition to the pure odd terms in $\mathbf{x}_0$. The contributions of the parametric excitation in the solvability condition (7) are

- For $n$ even the resonant terms are given for $m = 1, 3, \ldots, \; n-1, n+1$,
- For $n$ odd the resonant terms are given for $m = 2, 4, \ldots, \; n-1, n+1$.

In this case 2-period-QP and 3-period-QP can exist and periodic bursters are excluded.

- Case of parametric slow excitation

Periodic bursters can exist when $\mathbf{g}$ contains terms of the form $x^{2n+1}\cos(\varepsilon^2 t)$, otherwise QP solutions exist.

## 3. Applications

As a first example let us consider the following parametric damped equation:

$$\ddot{x} + x = h\cos(\tau)\dot{x} + \alpha\dot{x}^3. \tag{8}$$

Here $\tau = \varepsilon^2 t$, $h = \varepsilon\tilde{h}$ and $\alpha = \varepsilon\tilde{\alpha}$. Applying the MSM, the modulation equations of amplitude and phase are given by

$$\dot{a} = \frac{h}{2}a\cos(\tau) + \frac{3\alpha}{8}a^3, \tag{9}$$

$$\dot{\theta} = 0. \tag{10}$$

The solution of the modulation equation of the amplitude is given by

$$a(t) = \pm 2\frac{\sqrt{(-3\alpha e^{h\cos(\tau)t} + C4h\cos(\tau))h\cos(\tau)e^{h\cos(\tau)t}}}{-3\alpha e^{h\cos(\tau)t} + C4h\cos(\tau)}. \tag{11}$$

where $C$ is a constant of integration. The solution up to order $\mathcal{O}(\varepsilon)$ is given by

$$x(t) = a\cos(t+\theta) + \frac{\alpha}{4}a^3\sin(3t+3\theta) + O(\varepsilon^2). \tag{12}$$

where the nontrivial amplitude $a$ is given by

$$a = \sqrt{\frac{4h}{3\alpha}\cos(\tau)}. \tag{13}$$

The phase $\theta$ is a constant. The right hand side of Eq. (13) is imaginary when $\cos(\tau)$ is negative this means that the non-trivial amplitude does not exist. The nontrivial amplitude is stable when it exists and it coexists with a trivial amplitude that exists all the time. This latter is destabilized when the nontrivial amplitude exists. We conclude that there is existence of a periodic burster including a trivial and a periodic solution. In Fig. 1 the periodic burster is shown for $\alpha = -0.1$, $h = 0.1$. In all numerical computations the perturbation parameter $\varepsilon = 0.1$.

As a second example we consider a quasi-periodically parametric excited van der Pol equation

$$\ddot{x} + x = -\varepsilon[(\rho\cos(\nu t) + h\cos(\tau))x - \alpha\dot{x} + \beta x^2\dot{x}]. \tag{14}$$

Here $\tau = \varepsilon^2 t$. For details about this equation see [7,8]. Using the proposed method, the solution of the order $\mathcal{O}(1)$ is

$$x_0(T_0, T_1, \tau) = A(T_1, \tau)\exp(iT_0) + \text{cc}, \tag{15}$$

where cc denotes the complex conjugate of the preceding terms. The quantity $A(T_1, \tau)$ is to be determined by eliminating the secular terms at the next order of approximation.

### 3.1. Non-resonant case

The elimination condition of secular terms can be written as

$$i2(D_1A) = -hA\cos(\tau) + i(-\alpha A + \beta A^2\overline{A}). \tag{16}$$

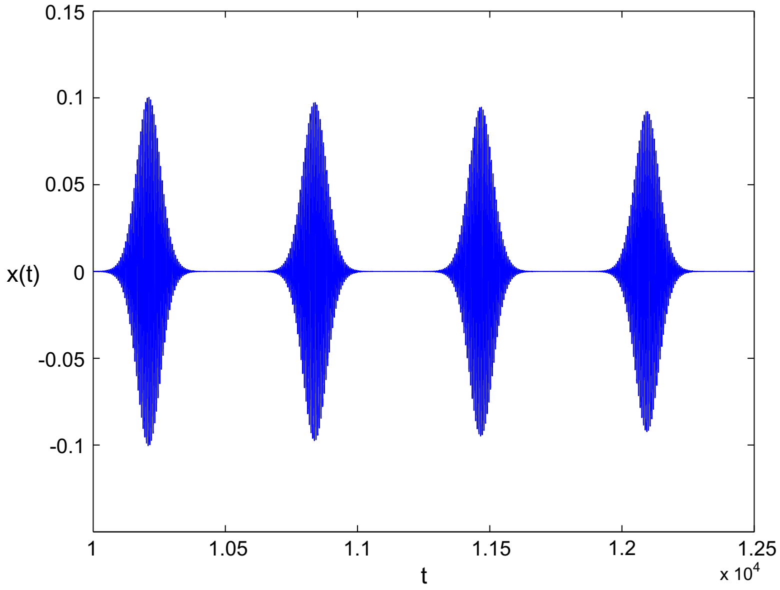


**Fig. 1.** Periodic burster of Eq. (8).

Let $A = \frac{1}{2}ae^{i\theta}$ where $a$ is the amplitude and $\theta$ is the phase. The imaginary part of Eq. (16) has two fixed points:

- Trivial solution that exists all the time and is stable when $\alpha > 0$, and unstable elsewhere,
- non-trivial solution $a_s = 2\sqrt{\alpha/\beta}$ which exists when $\alpha$ and $\beta$ have the same sign, and is stable only for $\alpha < 0$ and $\beta < 0$.

Fig. 2 shows zones of existence and stability of these solutions. It is to be pointed out that for $\alpha < 0$ and $\beta > 0$, the solution of Eq. (14) is unbounded for all initial conditions, since the unstable trivial solution is the only existing solution. The solution is unbounded also for $\alpha > 0$ and $\beta > 0$ but only for initial conditions greater than $a_s$, otherwise the stable trivial solution will attract the dynamics. For $\alpha > 0$ and $\beta < 0$ the solution is trivial. It is bounded and non-trivial only for $\alpha < 0$ and $\beta < 0$.

From the real part of Eq. (16), the modulated phase $\theta(\tau, T_1) = (h/2)\cos(\tau)T_1 + \theta_0$, where $\theta_0$ is a constant. An approximation of the solution of Eq. (14) is given by

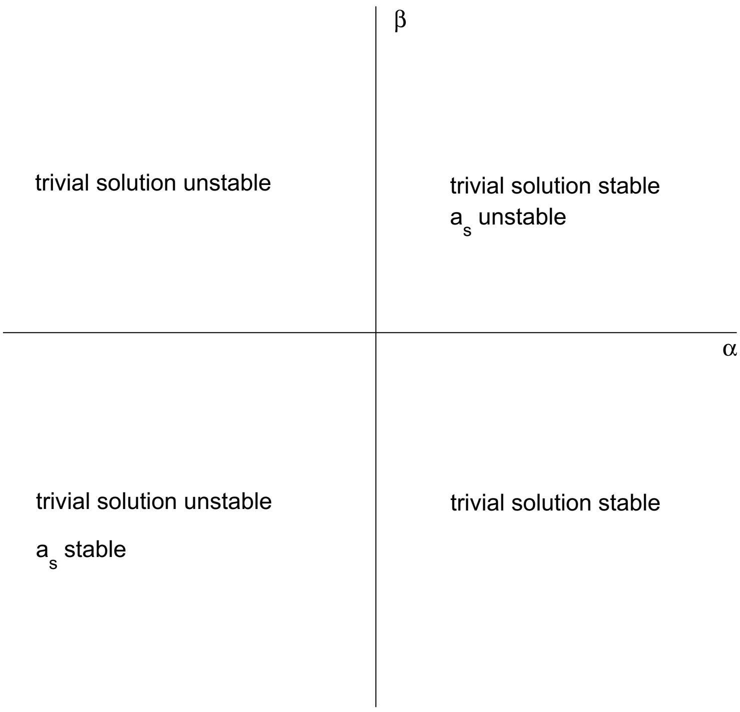


**Fig. 2.** Zones of existence and stability of the stationary amplitudes of Eq. (16) in the plane $\beta$ vs. $\alpha$.

$$x(t) = a_s\cos(t+\theta(\tau,T_1)) - \frac{\rho}{2(1-(\nu+1)^2)}a_s\cos((\nu+1)t+\theta(\tau,T_1)) - \frac{\rho}{2(1-(\nu-1)^2)}a_s\cos((\nu-1)t-\theta(\tau,T_1)) - \frac{\beta}{32}a_s^3\sin(3t+3\theta(\tau,T_1)) + \mathcal{O}(\varepsilon^2). \quad (17)$$

In this case there is no periodic bursters. For $\nu=\sqrt{2},\ \varepsilon=0.1,\ \rho=h=1$ and $\alpha=\beta=-1$, we show in Fig. 3 a comparison between power spectra of the numerical integration of Eq. (14) and the approximated solution (17). The power spectrum of $x(t)$ is defined as the absolute amplitude square of the Fourier transform of $x(t)$, it is a measurement of the power at various frequencies. It is carried out using the spectrum function in the signal processing toolbox of MATLAB. The sampling frequency is taken equal to 1000 Hz and the fourth-order Runge–Kutta method is used to obtain the numerical solution of Eq. (14). Note that the two power spectra are in good agreement.

### 3.2. Resonant case

We restrict our analysis to regular motions in the vicinity of the primary resonance one-half which is expressed as $\nu = 2+\varepsilon\sigma$. Eliminating secular terms leads to

$$i2(D_1A) = -\frac{\rho}{2}\overline{A}e^{i\sigma T_1} - hA\cos(\tau) + i(\alpha A - \beta A^2\overline{A}). \quad (18)$$

The modulation equations of amplitude $a$ and phase $\gamma = \frac{1}{2}\sigma T_1 - \theta$ are given by

$$\dot{a} = -\frac{\rho}{4\omega}a\sin(2\gamma) + \frac{\alpha}{2}a - \frac{\beta}{8}a^3 \quad (19)$$

$$\dot{\gamma} = \frac{\sigma}{2} - \frac{\rho}{4\omega}\cos(2\gamma) - \frac{h}{2\omega}\cos(\tau) \quad (20)$$

These equations have at most three fixed points and a limit cycle as stationary solutions.

- *Trivial solution:* $a=0$ is all the time unstable. It is an unstable focus when it is the only fixed point of Eq. (19), and an unstable node and/or a saddle when it coexists with non-trivial solutions.
- *Non-trivial quasi-static solutions:* $a_\pm(\tau)$

$$a_\pm^2 = \frac{8}{\beta}\left[\frac{\alpha}{2} \pm \sqrt{H(\tau)}\right], \quad \text{where} \quad H(\tau) = \frac{\rho^2}{16\omega^2} - \left(-\frac{\sigma}{2} + \frac{h}{2\omega}\cos(\tau)\right)^2. \quad (21)$$

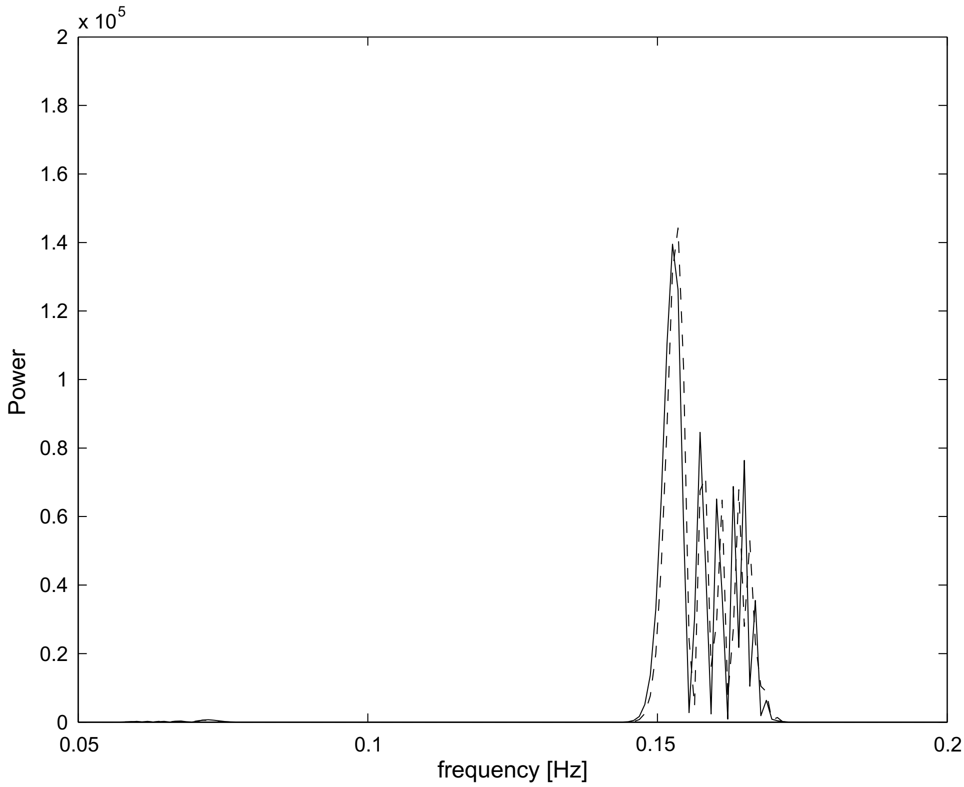


**Fig. 3.** Power spectrum: numerical integration of (14) in continuous line and the approximated solution (17)in dashed line.

These two solutions are periodic in $\tau$. The nontrivial solution $a_+$ exists if $H(\tau) \geqslant 0$ for all slow time $\tau$, this leads to the following condition of existence:

$$\rho \geqslant 4\omega \max\left(\left|-\frac{\sigma}{2} - \frac{h}{2\omega}\right|, \left|-\frac{\sigma}{2} + \frac{h}{2\omega}\right|\right) \quad (22)$$

The solution $a_+$ can exist partially during a period of the slow time scale $\tau$ for

$$4\omega \min\left(\left|-\frac{\sigma}{2} - \frac{h}{2\omega}\right|, \left|-\frac{\sigma}{2} + \frac{h}{2\omega}\right|\right) < \rho < 4\omega \max\left(\left|-\frac{\sigma}{2} - \frac{h}{2\omega}\right|, \left|-\frac{\sigma}{2} + \frac{h}{2\omega}\right|\right) \quad (23)$$

This equation defines the grey zone between regions I and II in Fig. 4. The condition of non existence of $a_+$ defines the region I in Fig. 4. The nontrivial solution $a_-$ exists for all $\tau$ if and only if $H(\tau) \geqslant 0$ and $(\alpha/2) - \sqrt{H(\tau)} \geqslant 0$ this leads to the following condition of existence:

$$4\omega \max\left(\left|-\frac{\sigma}{2} - \frac{h}{2\omega}\right|, \left|-\frac{\sigma}{2} + \frac{h}{2\omega}\right|\right) \leqslant \rho \leqslant 4\omega \min\left(\sqrt{\left(\frac{\alpha}{2}\right)^2 + \left(-\frac{\sigma}{2} + \frac{h}{2\omega}\right)^2}, \sqrt{\left(\frac{\alpha}{2}\right)^2 + \left(-\frac{\sigma}{2} + \frac{h}{2\omega}\right)^2}\right) \quad (24)$$

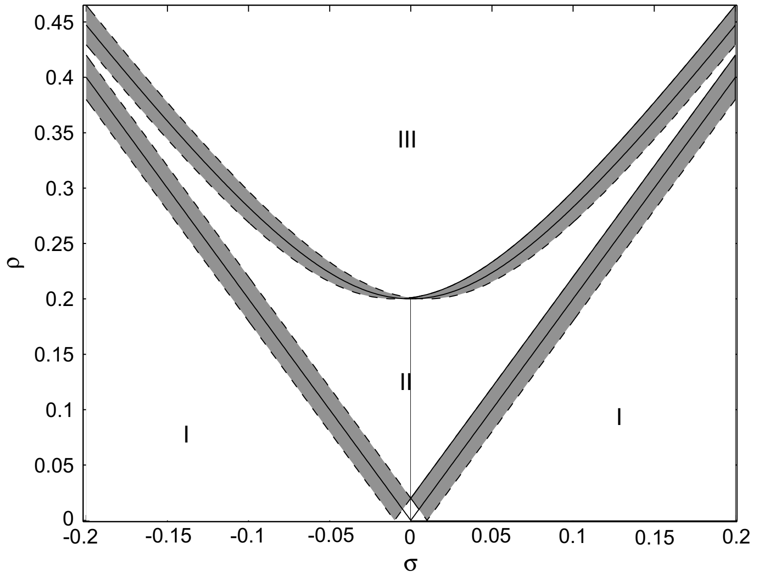


**Fig. 4.** Chart of behaviors of modulation Eqs. (19) and (20) for $\alpha = \beta = 1$. In zone I, existence of a limit cycle. In zone II, coexistence of a stable solution $a_+$ and an unstable solution $a_-$. In zone III, existence of the stable solution $a_+$. The grey zones correspond to the zones where periodic bursters can exist. In the absence of the slow driver i.e., $h = 0$ the chart is indicated by solid lines passing through the grey zones.

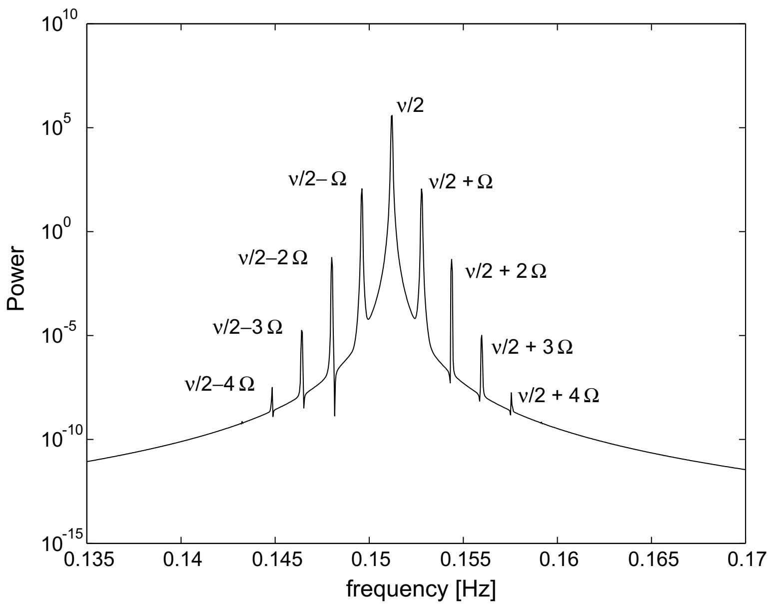


**Fig. 5.** Power spectrum corresponding to Eq. (14) for $\sigma = -1$ and $\rho = 4$.

This equation defines region II in Fig. 4. It is worth noting that the solution $a_+$ is stable when it exists and $a_-$ is unstable when it exists.

- *Limit cycle:* It exists and it is stable in the regions of parameters where the non-trivial solutions $a_\pm$ do not exist [8], i.e., region I in Fig. 4.

In Fig. 4 is shown the chart of behaviors of modulation equations of amplitude (19). The solid lines passing through the grey zones correspond to the chart in the absence of the slow driver i.e., $h = 0$. The grey zones exist only for $h \neq 0$ and they correspond to the zones where periodic bursters can exist. In the zone I exists a limit cycle. In zone II, coexistence of $a_+$ and $a_-$. In zone III, existence of only $a_+$. The grey zone between I and II corresponds to a periodic burster relating the limit cycle to $a_+$. The grey zone between II and III, does not contain a periodic burster, it corresponds to the fact that $a_-$ does not exist during all the time during a period $2\pi/\varepsilon^2$.

The approximated solution of Eq. (14) is given by

$$x(t) = a(\tau)\cos\left(\frac{v}{2}t - \gamma(\tau)\right) - \frac{\rho}{2(1-(v+1)^2)}a(\tau)\cos\left(\frac{3v}{2}t - \gamma(\tau, T_1)\right) - \frac{\beta}{32}a^3(\tau)\sin\left(\frac{3v}{2}t - 3\gamma(\tau)\right) + \mathcal{O}(\varepsilon^2). \tag{25}$$

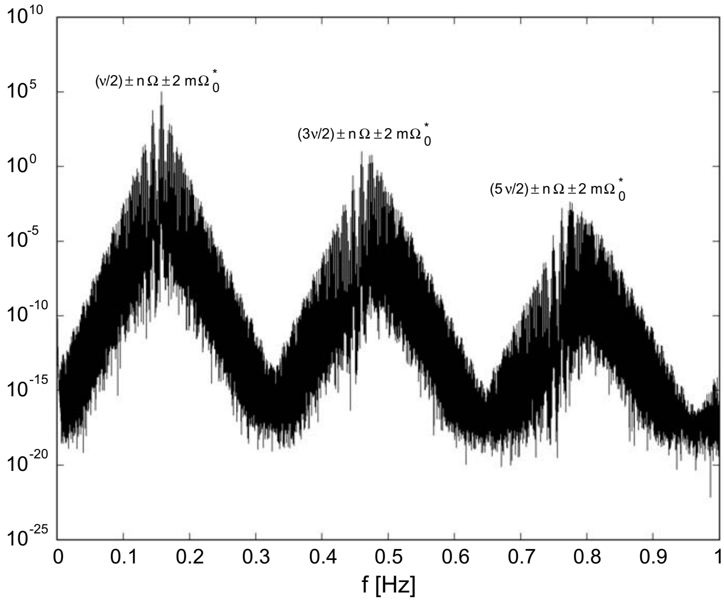


**Fig. 6.** Power spectrum corresponding to Eq. (14) $\sigma = -1$ and $\rho = 1$.

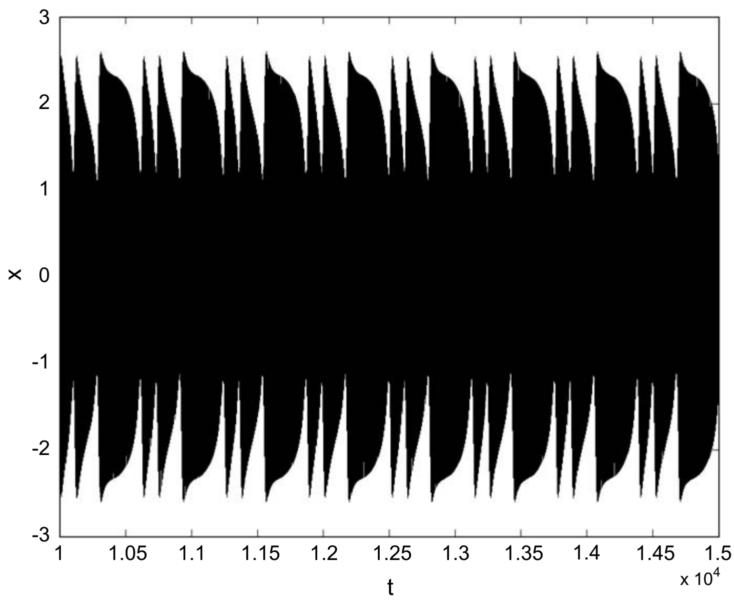


**Fig. 7.** Periodic burster solution involving 2-period-QP solution and 3-period-QP solution, for $\rho = 1.85$ and $v = 1.9$.

Using the approximated solution (25), the chart of behaviors of Eq. (14) can be deduced from Fig. 4. Thus the solution in the zones II and III is 2-period-QP with fundamental frequencies $\nu/2$ and $\varepsilon^2$ (see Fig. 5). In the zone I, the solution is 3-period-QP solution with $(\nu/2)$, $\varepsilon^2$ and $2\Omega_0^*$ as fundamental frequencies (see Fig. 6). Here $\Omega_0^*$ is the frequency of the self-excited limit cycle. In the grey zones a given solution changes its stability or ceases to exist in a part of the period of the very slow dynamics. In the grey region between the zone I and II, the periodic burster solution is a heteroclinic connection between the 2-period-QP solution and the 3-period-QP solution (see Fig. 7). The grey region between the zone II and III does not involve any change for the stable 2-period-QP solution. It involves only the disappearance of the unstable 2-period-QP one.

## 4. Conclusion

In this paper, we studied the occurrence of periodic bursters and QP solutions of quasi-periodically excited oscillators. The QP forcing consists of a resonant frequency with the proper frequency of the oscillator and a very slow frequency. An averaging of the fast dynamics showed that a slow parametric excitation is necessary for the existence of periodic bursters. The existence conditions of QP solutions are discussed and charts of behaviors are plotted.

As a continuation of this study more oscillators will be investigated and the nonregular behaviors will be studied.

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## References

[1] Broer H, Simó C. Hill's equation with quasi-periodic forcing:resonance tongues, instability pockets and global phenomena. Bull Soc Bras Mat 1998:253.
[2] Rand R, Zounes R, Hastings R. Dynamics of a quasiperiodically forced Mathieu oscillator. In: Guran A, editor. Non-linear dynamics: The Richard Rand 50th anniversary volume. Singapore: World Scientific; 1997. p. 203–21.
[3] Belhaq M, Guennoun K, Houssni M. Asymptotic solutions for a damped non-linear quasi-periodic Mathieu equation. Int J Nonlinear Mech 2002;37:445–60.
[4] Belhaq M, Houssni M. Quasi-periodic oscillations chaos and suppression of chaos, in a nonlinear oscillator driven by parametric and external excitations. Nonlinear Dyn 1999;18:1–24.
[5] Golubitsky M, Josic K, Kaper TJ. An unfolding theory approach to bursting in fast–slow systems. In: Broer H, Krauskopf B, Vegter G, editors. Global analysis of dynamical systems. Bristol: IOP; 2001.
[6] Nayfeh AH. Perturbation methods. New York: Wiley; 1973.
[7] Abouhazim N, Belhaq M, Lakrad F. Three-period quasi-periodic solutions in the self-excited quasi-periodic Mathieu oscillator. Nonlinear Dyn 2005;39:395–409.
[8] Lakrad F, Azouani A, Abouhazim N, Belhaq M. Bursters and quasi-periodic solutions of a self-excited quasi-periodic Mathieu oscillator. Chaos Soliton Fract 2005;24:813–24.