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ANALYTICAL PREDICTION OF THE TWO FIRST PERIOD-DOUBLINGS IN A THREE-DIMENSIONAL SYSTEM

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Analytical study of the two first period-doubling bifurcations in a three-dimensional system is reported. The multiple scales method is first applied to construct a higher-order approximation of the periodic orbit following Hopf bifurcation. The stability analysis of this periodic orbit is then performed in terms of Floquet theory to derive the critical parameter values corresponding to the first and second period-doubling bifurcations. By introducing suitable subharmonic components in the first order of the multiple scale analysis the two critical parameter values are obtained simultaneously solving analytically the resulting system of two algebraic equations. Comparisons of analytic predictions to numerical simulations are also provided.

1. Introduction

In recent years, there has been a great interest in understanding the dynamics and bifurcations in problems governed by three-dimensional autonomous differential systems. A book has been recently devoted to such problems showing several kinds of three-dimensional models with interesting applications in astronomy, see [Kandrup *et al.*, 1995]. In contrast with the two-dimensional systems, widely studied over this century, the three-dimensional systems are still largely unexplored, specially from the analytical point of view. Indeed, few analytical techniques in terms of bifurcation analysis have been developed so far. Even an analytical approximation of a periodic solution is impossible to construct in some three-dimensional systems [Hale & Koçak, 1991]. However the numerical experiment of such systems by means of software continuation codes has received more and more efforts.

The dynamics of three-dimensional systems near one of their periodic orbits is very rich in terms of bifurcation and stability. Important behaviors include symmetry-breaking, period-doubling, Neimark–Sacker (bifurcation to invariant torus) and specially homoclinic or Shil'nikov bifurcation. Recent works has been focused on analytical investigations of symmetry-breaking and period-doubling bifurcations (see [Rand, 1989; Belhaq & Houssni, 1995] for the system (1) and [Phillipson & Schuster, 1998] for the system introduced by Arneodo *et al.* [1985]).

The purpose of the present work is the analytical study of the first and second period-doubling bifurcations of a periodic orbit born in a Hopf bifurcation of the three-dimensional system

$$\dot{x} = \mu x - y - xz\,,$$
$$\dot{y} = \mu y + x\,, \tag{1}$$
$$\dot{z} = -z + x^2 z + y^2\,.$$

The system (1) may be thought of as a response control system consisting of a damped linear oscillator in x, y variables and a control variable z. The "dot" denotes time derivative, x, y and z are scalar variables and μ is a scalar parameter. The origin ($x_0 = 0$, $y_0 = 0$, $z_0 = 0$) of system (1) is an equilibrium, stable for $\mu < 0$ and unstable for $\mu > 0$ so that a Hopf bifurcation occurs at $\mu = 0$. As the parameter μ increases from zero, the periodic orbit undergoes a symmetry-breaking bifurcation at $\mu = \mu_{\text{SB}}$. As μ increases again, this orbit becomes unstable and a new stable periodic orbit of twice the period appears by period-doubling at $\mu = \mu_{\text{PD1}}$. This orbit undergoes a second period-doubling bifurcation at $\mu = \mu_{\text{PD2}}$. A brief numerical study is given in Sec. 5.

Using the center manifold theory and a near-identity transformation, Rand and Armbruster [1987] and Rand [1989] constructed a first-order approximation of the limit cycle near the Hopf bifurcation. The critical value $\mu_{\text{PD1}}(\approx 0.45)$ corresponding to the first period-doubling bifurcation was approached by studying the stability of the orbit in direction of z normal to the center manifold. Nayfeh and Balachandran [1990] used the method of multiple scales [Nayfeh, 1973] to obtain, as in [Rand, 1989], the same first-order approximation of the periodic solution. The critical values $\mu_{\text{SB}}(\approx 0.30)$ and $\mu_{\text{PD1}}(\approx 0.4405)$ were approximated numerically using the Floquet theory [Nayfeh & Mook, 1979].

In [Belhaq & Houssni, 1995] a higher-order approximation of the periodic orbit of system (1) was constructed using a higher-order multiple scales expansion. A stability analysis was performed to predict approximations of the critical parameter values $\mu_{\text{SB}}(\approx 0.31)$ and $\mu_{\text{PD1}}(\approx 0.446)$.

In a recent work Belhaq *et al.* [1999] focused attention on the analytical prediction of a homoclinic bifurcation occuring in system (1). The approximation to the critical parameter value of this bifurcation was derived using the so-called collision criterion (see [Belhaq *et al.*, 1999a, 1999b]).

For another three-dimensional system, Phillipson and Schuster [1998] applied an asymptotic averaging formalism to derive approximate analytic expressions for the location of the first period-doubling bifurcation point (period-one to period-two bifurcations) for the system introduced by Arneodo *et al.* [1985]. The method applied formally links this bifurcation with similar behavior in the two-dimensional Duffing's equation driven by periodic external forcing. Donescu and Virgin [1996] considered the case of second-order differential equation with periodic forcing and used the method of harmonic balance with an arbitrary number of modes in the assumed solution to determine the set of algebraic equations. Numerical methods have been employed to solve this algebraic system in terms of computing the three first period-doubling bifurcations. Maggio *et al.* [1998] studied the first period-doubling bifurcation in the Colpitts oscillator (a piecewise-linear system) by exploiting harmonic balance technique, as in [Basso *et al.*, 1997], for solving the variational equation associated with a generic limit cycle. Note that similar strategy was adressed in [Belhaq & Houssni, 1995] for predicting symmetry-breaking in system (1).

In this work we derive an analytical approximation of the critical values μ_{PD1} and μ_{PD2} corresponding to the two first period-doubling bifurcations of the system (1) using the multiple scales technique and a stability analysis as in [Rand, 1989] and [Belhaq & Houssni, 1995]. By introducing suitable subharmonic components to the T-periodic solution at the first-order step of system of the multiple scale analysis, the two critical values μ_{PD1} and μ_{PD2} are obtained simultaneously solving analytically two resulting algebraic equations.

This paper is organized as follows. In Sec. 2 we derive a higher-order approximation of the periodic orbit using the multiple scales technique. In Sec. 3 the symmetry-breaking bifurcation is briefly discussed performing the stability analysis of the periodic solution via Floquet theory and the method of harmonic balance. Section 4 is devoted to the investigation of the period-doubling bifurcations. We perform numerical simulation in Sec. 5. Finally we conclude with a discussion of the results.

2. Asymptotic Expansion

Using the method of multiple scales [Nayfeh, 1973], a higher-order uniformally valid asymptotic expansion for the periodic orbit of the system (1) may be sought in the form

$$x = \sum_{n=0}^{6} \varepsilon^n x_n(T_0, T_1, T_2, T_3, T_4, T_5) + \cdots\,, \tag{2}$$

$$y = \sum_{n=0}^{6} \varepsilon^n y_n(T_0, T_1, T_2, T_3, T_4, T_5) + \cdots ,$$

$$z = \sum_{n=0}^{6} \varepsilon^n z_n(T_0, T_1, T_2, T_3, T_4, T_5) + \cdots ,$$

where $T_n = \varepsilon^n t$ are the time scales and ε is a small positive dimensionless parameter assumed to be of the order of the amplitude of the motion. The control parameter is expanded as $\mu = \varepsilon^2\mu_2 + O(\varepsilon^3)$.

Substituting this last relation and Eqs. (2) into (1), taking into account $x_0 = 0$, $y_0 = 0$, $z_0 = 0$, and equating coefficients of like powers of ε, we obtain at different orders of ε the following systems of successive approximations x_n, y_n, z_n

$$o(\varepsilon^1): \quad \begin{aligned} D_0x_1 + y_1 &= 0\,, \\ D_0y_1 - x_1 &= 0\,, \\ D_0z_1 + z_1 &= 0\,. \end{aligned} \tag{3}$$

$$o(\varepsilon^i,\, i \geq 2): \quad \begin{aligned} D_0x_i + y_i &= \mu_2 x_{i-2} - \sum_{j=0}^{i} x_j z_{i-j} - \sum_{j=1}^{i} D_j x_{i-j}\,, \\ D_0y_i - x_i &= \mu_2 y_{i-2} - \sum_{j=1}^{i} D_j y_{i-j}\,, \\ D_0z_i + z_i &= \sum_{j=0}^{i} y_j y_{i-j} - \sum_{j=1}^{i} D_j z_{i-j} + \sum_{j=0}^{i}\left(x_{i-j}\sum_{k=0}^{j} z_k x_{j-k}\right)\,, \end{aligned} \tag{4}$$

where $D_n = \partial/\partial T_n$. The explicit expression to higher-order of the system (4) are detailed in [Belhaq & Houssni, 1995]. Using Eqs. (3) and (4) for $i = 2$, 3, the solution up to the second order is given by [Nayfeh & Balachandran, 1990]

$$\begin{aligned} x(t) &= -\varepsilon a \sin\theta + O(\varepsilon^3)\,, \\ y(t) &= \varepsilon a \cos\theta + O(\varepsilon^3)\,, \\ z(t) &= \frac{1}{2}\varepsilon^2 a^2\left[1 + \frac{1}{5}\cos(2\theta) + \frac{2}{5}\sin(2\theta)\right] + O(\varepsilon^3)\,, \end{aligned} \tag{5}$$

where

$$a = \frac{2}{3}\sqrt{10\mu_2}, \quad \theta = \left(1 - \frac{\varepsilon^2 a^2}{20}\right)t + O(\varepsilon^3)\,. \tag{6}$$

These values of a and θ are obtained from the two conditions

$$D_1A = 0, \quad 2D_2A - 2\mu_2 A + \frac{9+2i}{5}A^2\overline{A} = 0\,, \tag{7}$$

that vanish the secular terms in (4), for $i = 2$ and $i = 3$. Now let us investigate a higher-order approximation of the limit cycle. The general solution of the system (4) for $i = 3$ is given by

$$\begin{aligned} x_3 &= \frac{3}{40}(2i-1)A^3e^{3iT_0} + \text{c.c}\,, \\ y_3 &= \frac{2+i}{40}A^3e^{3iT_0} - \frac{i(9+2i)}{10}A^2\overline{A}e^{iT_0} + \text{c.c}\,, \\ z_3 &= 0\,, \end{aligned} \tag{8}$$

where c.c stands for the complex conjugate of the preceding expressions. Hence, from Eqs. (5)–(8), the approximation of the periodic solution to third order is

$$\begin{aligned} x(t) &= -\varepsilon a \sin\theta - \frac{3\varepsilon^3 a^3}{80}\left(\frac{1}{2}\cos(3\theta) + \sin(3\theta)\right) + O(\varepsilon^4)\,, \\ y(t) &= \varepsilon a \cos\theta + \frac{\varepsilon^3 a^3}{20}\left(\cos\theta + \frac{9}{2}\sin\theta + \frac{1}{4}\cos(3\theta) - \frac{1}{8}\sin(3\theta)\right) + O(\varepsilon^4)\,, \\ z(t) &= \frac{\varepsilon^2 a^2}{2}\left(1 + \frac{1}{5}\cos(2\theta) + \frac{2}{5}\sin(2\theta)\right) + O(\varepsilon^4)\,, \end{aligned} \tag{9}$$

where a and θ are given by Eqs. (6).

On the other hand, the systems (4) for $i = 4, 5, 6$ allow one to determine the approximation of the periodic orbit up to the fifth order. Indeed, the elimination of the secular terms in the system (4) for $i = 4$ leads to the condition

$$D_3 A = 0\,, \tag{10}$$

and then the solution up to the fourth order can be written as

$$\begin{aligned} x_4 &= 0\,,\\ y_4 &= 0\,,\\ z_4 &= \frac{9i-2}{20(1+4i)}A^4 e^{4iT_0} - \frac{2}{(1+2i)^2}AD_2Ae^{2iT_0}\\ &\quad - \frac{22+51i}{20(1+2i)}A^3\overline{A}e^{2iT_0} + \frac{11-7i}{5}(A\overline{A})^2\\ &\quad - D_2(A\overline{A}) + \text{c.c.} \end{aligned} \tag{11}$$

The equation that eliminates the secular terms in (4) for $i = 5$ is given by the condition:

$$D_4 A = \mu_2 K A^2 \overline{A} + H A^3 (\overline{A})^2\,, \tag{12}$$

where

$$K = \frac{38}{25} + \frac{61}{100}i, \quad H = -\frac{3052+861i}{2000}\,. \tag{13}$$

Similarly we obtain from the higher-order system the last condition vanishing secular terms in Eq. (4) for $i = 6$

$$D_5 A = 0\,. \tag{14}$$

Therefore the set of five conditions to be resolved is given by

$$D_1 A = 0, \tag{15}$$

$$2D_2A - 2\mu_2 A + \frac{9+2i}{5}A^2\overline{A} = 0, \tag{16}$$

$$D_3 A = 0, \tag{17}$$

$$D_4 A = \mu_2\left(\frac{38}{25}+\frac{61}{100}i\right)A^2\overline{A} - \frac{3052+861i}{2000}A^3(\overline{A})^2, \tag{18}$$

$$D_5 A = 0. \tag{19}$$

Substituting $A = 1/2ae^{i\beta}$ (where a and β are real quantities) into Eqs. (15)–(19), separating real and imaginary parts, we obtain

$$D_2 a = \mu_2 a - \frac{9}{40}a^3\,, \tag{20}$$

$$D_4 a = \frac{19}{50}\mu_2 a^3 - \frac{763}{8000}a^5\,, \tag{21}$$

$$D_4 \beta = -\frac{1}{20}a^2\,, \tag{22}$$

$$D_4 \beta = \frac{61}{400}\mu_2 a^2 - \frac{861}{32000}a^4\,. \tag{23}$$

The solution of Eq. (20) is

$$a = C(T_4)\sqrt{\mu_2 - \frac{9}{40}a^2}\exp(\mu_2 T_2)\,, \tag{24}$$

where a and μ_2 satisfy the condition

$$a^2 < \frac{40}{9}\mu_2\,. \tag{25}$$

Solving Eq. (24) for a yields

$$a = \frac{C(T_4)\sqrt{\mu_2}\exp(\mu_2 T_2)}{\sqrt{1+\frac{9}{40}(C(T_4))^2\exp(2\mu_2 T_2)}}\,, \tag{26}$$

where $C(T_4)$ is an arbitrary function of T_4. Now the substitution of Eq. (26) into Eq. (21) provides an ordinary differential equation on $C(T_4)$ of the form

$$\begin{aligned} &\frac{1+\frac{9}{40}(C(T_4))^2\exp(2\mu_2 T_2)}{\frac{19}{50}(C(T_4))^3 - \frac{79}{8000}(C(T_4))^5\exp(2\mu_2 T_2)}dC\\ &\quad = \mu_2^2\exp(2\mu_2 T_2)dT_4\,, \end{aligned} \tag{27}$$

in which T_2 is to be treated as a constant. This may be integrated to give

$$\begin{aligned} &(C(T_4))^2\left[\exp\left(-\frac{\sigma}{(C(T_4))^2\alpha}\right) + E\exp\left(FT_4 + \frac{\text{p.c}}{\alpha}\right)\right]\\ &\quad = \frac{19}{50}\exp\left(FT_4 + \frac{\text{p.c}}{\alpha}\right)\,, \end{aligned} \tag{28}$$

where

$$E = \frac{79}{8000}\exp(2\mu_2 T_2), \quad F = \frac{2\mu_2^2}{\alpha}\exp(2\mu_2 T_2)\,,$$

$$\sigma = \frac{50}{19}, \quad \alpha = \frac{763}{3040}\sigma\exp(2\mu_2 T_2) \tag{29}$$

and p.c denotes a pure constant. This last equation is a transcendental one for $C(T_4)$, and then one cannot obtain a closed form expression for $C(T_4)$ to be substituted into the previous expression (26) for a. Nevertheless, in order to overcome this difficulty, it turns out that an approached solution of $C(T_4)$ can be obtained by expanding the function $\exp(-\sigma/(C(T_4))^2\alpha)$ and retaining only the two first terms of the expansion. Expanding, substituting these terms in Eq. (28) and resolving in $C(T_4)$, we obtain an approximation of $C(T_4)$ as follows

$$C(T_4) = \left[\frac{\dfrac{\sigma}{\alpha} + \dfrac{19}{50}\exp\left(FT_4 + \dfrac{\text{p.c}}{\alpha}\right)}{1 + E\exp\left(FT_4 + \dfrac{\text{p.c}}{\alpha}\right)}\right]^{\frac{1}{2}} . \qquad (30)$$

Hence the previous expression of a, given by Eq. (26), becomes

$$a = \frac{\left[\mu_2\left(\dfrac{3040}{763}\exp(-2\mu_2 T_2) + \dfrac{19}{50}\exp\left(2\mu_2 T_2 + \dfrac{11552}{3815}\mu_2^2 T_4 + \dfrac{\text{p.c}}{\alpha}\right)\right)\right]^{\frac{1}{2}}}{\left[\dfrac{1447}{763} + \dfrac{763}{8000}\exp\left(2\mu_2 T_2 + \dfrac{11552}{3815}\mu_2^2 T_4 + \dfrac{\text{p.c}}{\alpha}\right)\right]^{\frac{1}{2}}} . \qquad (31)$$

It follows that the new approximation of the amplitude of the limit cycle is $a \to 20\sqrt{(38/3815)\mu_2}$ as $t \to \infty$. This expression verifies, as expected, the condition given by Eq. (25). Note that the first approximation of the amplitude a is given by Eq. (6). Consequently, from Eqs. (9) and (11), and the solution of Eqs. (4) for $i = 5$, we obtain the following fifth-order approximation of the periodic solution

$$\begin{aligned} x(t) &= -\varepsilon a \sin\theta - \frac{3\varepsilon^3 a^3}{80}\left(\frac{1}{2}\cos(3\theta) + \frac{1}{2}\sin(3\theta)\right) \\ &\quad + \frac{\varepsilon^5 a^5}{5120}\left(-\frac{32}{3}\cos(5\theta) + \frac{1}{3}\sin(5\theta) + \frac{11587}{95}\cos(3\theta) - \frac{3247}{190}\sin(3\theta)\right) + O(\varepsilon^6)\,, \end{aligned} \qquad (32)$$

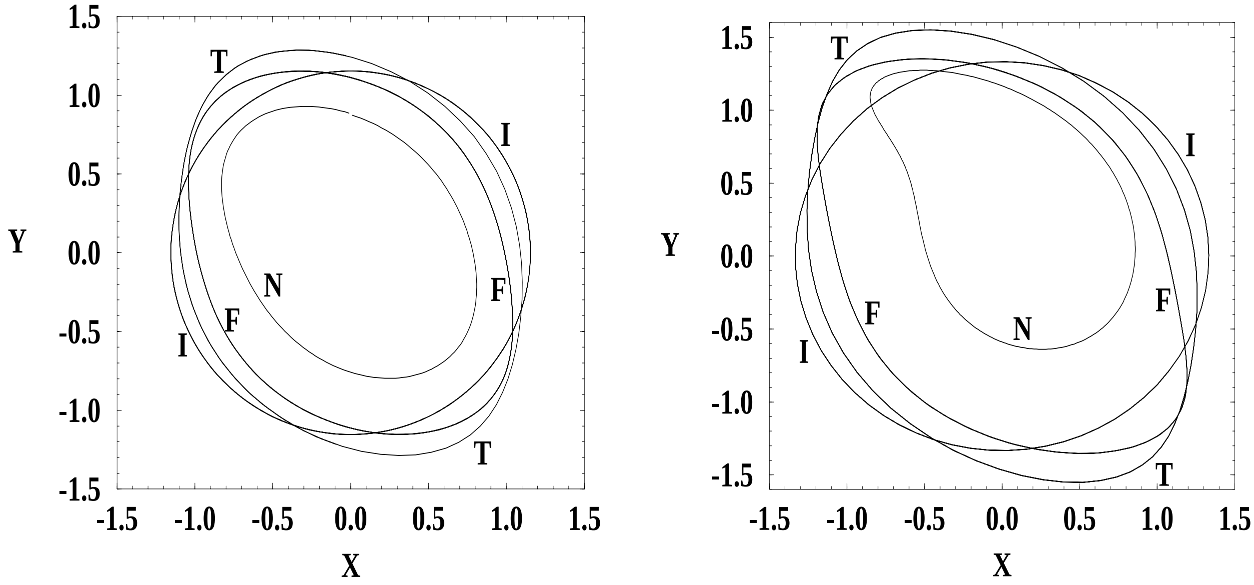
$$\begin{aligned} y(t) &= \varepsilon a \cos\theta - \frac{\varepsilon^3 a^3}{20}\left(\cos\theta + \frac{9}{2}\sin\theta + \frac{1}{4}\cos(3\theta) - \frac{1}{8}\sin(3\theta)\right) \\ &\quad + \frac{\varepsilon^5 a^5}{1600}\left(-\frac{197771}{2888}\cos\theta + \frac{1289}{19}\sin\theta + \frac{186181}{69312}\cos(3\theta)\right. \\ &\quad \left. - \frac{180177}{103968}\sin(3\theta) - \frac{1}{48}\cos(5\theta) - \frac{2}{3}\sin(5\theta)\right) + O(\varepsilon^6)\,, \end{aligned} \qquad (33)$$

$$\begin{aligned} z(t) &= \frac{\varepsilon^2 a^2}{2}\left(1 + \frac{1}{5}\cos(2\theta) + \frac{2}{5}\sin(2\theta)\right) \\ &\quad + \frac{\varepsilon^4 a^4}{80}\left(\frac{757}{38} - \frac{2187}{190}\cos(2\theta) + \frac{161}{190}\sin(2\theta) + \cos(4\theta) - \frac{1}{2}\sin(4\theta)\right) + O(\varepsilon^6)\,, \end{aligned} \qquad (34)$$

where a and θ are now given by the new approximations

$$\begin{aligned} a &= 20\sqrt{\frac{38}{3815}\mu_2}\,, \\ \theta &= \left(1 - \frac{1}{20}\varepsilon^2 a^2 + \frac{2765}{243200}\varepsilon^4 a^4\right)t + O(\varepsilon^6)\,. \end{aligned} \qquad (35)$$

In Fig. 1 we compare the approximations at different orders of the periodic orbit obtained by the multiple-scales technique, Eqs. (32)–(35), with the periodic orbit obtained numerically by integrating the system (1) for two parameter values of μ. For a survey regarding higher-order approximations for the periodic solutions using the harmonic balance method, in general nonlinear dynamical systems, see [Moiola & Chen, 1996].



(a) (b)

Fig. 1. Comparison of different approximations of periodic orbits for: (a) $\mu = 0.3$; (b) $\mu = 0.4$. Label N corresponds to the exact orbit obtained by numerical integration. I indicates first-order approximation, T denotes the third-order approximation orbit and F corresponds to orbit obtained with the fifth-order approximation.

3. Symmetry-Breaking Bifurcation

In this section we briefly discuss the methodology followed to predict analytically a closer approximation of the symmetry-breaking bifurcation. Since the symmetry occurs in the (x, y)-plane, the conditions to be satisfied are $x(t) = -x(t + (T/2))$, $y(t) = -y(t + (T/2))$ where T is the period of the orbit. Following Szemplinska-Stupnicka [1990], the stability analysis of this orbit may be carried out by disturbing the system (1) in the (x, y)-plane as

$$x(t) = x_A + \delta x(t), \quad y(t) = y_A + \delta y(t), \qquad (36)$$

where x_A and y_A denote the third-order approximation of the periodic orbit given by Eqs. (9).

Note that Hassan [1996] analyzed the local stability of the approximate solution using harmonic balance to harmonically excited nonlinear oscillators. It was shown that a consistent stability type information about these solutions can be provided only when linearized variational equation is analyzed by approximate methods, and the level of accuracy of this analysis is consistent with that of the approximate solutions. Otherwise the stability analysis of the nonlinear variational equation can provide erroneous results when using the "steady state" harmonic balance.

In our case, a three-dimensional system, we use the approximation of the solution obtained by the "transient" multiple scales method and perform the stability analysis. The higher harmonics used in the solution and in the stability analysis are in the same level of approximation (third-order) for both the symmetry-breaking and the period-doubling stability analysis (see [Belhaq & Houssni, 1995]). This procedure, applied in harmonically excited nonlinear oscillators seems to give qualitatively correct predictions of the critical parameter value of bifurcations in a three-dimensional system.

Substituting Eqs. (36) into Eq. (1), linearizing in the variation δx, δy and expanding x_A and y_A into Fourier series, we obtain the variational equation

$$\delta\dot{x} = (\lambda_0^z + \lambda_{2c}^z \cos(2\beta) + \lambda_{2s}^z \sin(2\beta))\delta x - \delta y,$$
$$\delta\dot{y} = \delta x + \mu\delta y, \qquad (37)$$

where

$$\lambda_0^z = \mu - \frac{a^2}{2}, \quad \lambda_{2c}^z = \frac{a^2}{10}, \quad \lambda_{2s}^z = \frac{a^2}{5}. \qquad (38)$$

Using Floquet theory, the particular solution of Eqs. (37) can be sought in the form

$$\delta x(t) = \exp(\varepsilon_1 t)\Phi_1(t),$$
$$\delta y(t) = \exp(\varepsilon_2 t)\Phi_2(t), \qquad (39)$$

where $\Phi_i(t)$ is a periodic function of time related to the period T. The condition that allows us to investigate the symmetry-breaking bifurcation (see [Szemplinska-Stupnicka, 1990]) can be written in the form

$$\Phi_i(t) = \Phi_i\left(t + \frac{T}{2}\right), \quad \varepsilon_i > 0, \quad i = 1, 2. \tag{40}$$

Expanding $\Phi_i(t)$ into Fourier series yields

$$\Phi_i(t) = b_0^i + b_2^i \cos(2\beta + \varphi) + \sum_{n=4,6,\ldots}^{k} b_n^i \cos(n\beta + \varphi), \quad i = 1, 2. \tag{41}$$

As in the forced oscillators cases [Nayfeh & Mook, 1979; Szemplinska-Stupnicka, 1990], the bifurcation of the symmetric solution into the unsymmetric one can occur by instability phenomena resulting from a growth of even order harmonics. To study this instability we assume that

$$\begin{aligned} \delta x(t) &= \exp(\varepsilon_1 t)\left(b_0^1 + b_2^1 \cos(2\beta + \varphi) + \sum_{n=4,6,\ldots} b_n^1 \cos(n\beta + \varphi)\right), \\ \delta y(t) &= \exp(\varepsilon_2 t)\left(b_0^2 + b_2^2 \cos(2\beta + \varphi) + \sum_{n=4,6,\ldots} b_n^2 \cos(n\beta + \varphi)\right). \end{aligned} \tag{42}$$

Substituting Eqs. (42) into Eqs. (37) and applying the harmonic balance method yield the set of four equations

$$(M)\begin{bmatrix} b_0^1 \\ b_0^2 \\ b_2^1 \\ b_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad (M) = \begin{pmatrix} \lambda_0^z & -1 & 0 & 0 \\ 1 & \mu & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{43}$$

Hence, the condition to be satisfied at the stability limit is given by vanishing the characteristic determinant of the system (43). A conjecture established in [Belhaq & Houssni, 1995] leads to an analytical approximation of $\mu_{\mathrm{SB}} (\approx 0.31)$ which is in good agreement with the numerical study performed in Sec. 5 of the present paper.

4. First and Second Period-Doubling Bifurcations

Using the center manifold theory and introducing a near-identity transformation, Rand [1989] determined an approximation of the center manifold and of the limit cycle to first order near Hopf bifurcation. He focused on the last equation of system (1) which governs the stability of the periodic orbit in terms of period-doubling. Indeed, the period-doubling cannot take place as long as the periodic orbit lies in the two-dimensional center manifold, otherwise trajectories self-intersect. The approximation was used to investigate the stability of the limit cycle by linearizing in the variation of δz as $z = z_{lc} + \delta z$. This is

$$\delta\dot{z} = (-1 + x_{lc}^2)\delta z, \tag{44}$$

where x_{lc} denotes the first-order approximation of the limit cycle in x. This equation has the general solution

$$\delta z(t) = \delta z(0) \exp\left(\int_0^t \left(-1 + \frac{20}{9}\mu + \frac{20}{9}\mu\right)\cos(2\theta) d\theta\right). \tag{45}$$

The transition from stable to unstable solution occurs when the condition $\delta z(T) = \delta z(0)$ is satisfied, where T is the period of the limit cycle (see [Rand, 1989]). Therefore the critical parameter value of μ is given, to the first-order approximation, by

$$-1 + \frac{20}{9}\mu_{\mathrm{PD1}} = 0, \tag{46}$$

which provides an approximation of the critical parameter value of the first period-doubling bifurcation $\mu_{\mathrm{PD1}} = 0.45$. In [Belhaq & Houssni, 1995] a third-order approximation of the limit cycle was performed. This approximation was introduced in

Eq. (45) to obtain a third-order algebraic equation in $\mu_{\rm PD1}$ [instead of Eq. (46)] leading to an improved approximation of the critical value of $\mu_{\rm PD1}$ (≈ 0.446) (see Sec. 5 for comparison to numerical simulations).

Now we shall focus attention on the prediction of the second period-doubling bifurcation. To do so, we specifically develop a strategy to construct an analytical approximation of the critical values $\mu_{\rm PD1}$ and $\mu_{\rm PD2}$ simultaneously. The idea consists in introducing the required subharmonic terms into the periodic solution at the first-order multiple scale expansion involving suitable components in the periodic solution. In this context, the solution in the vicinity of the period-doubling branch is postulated as

$$
\begin{aligned}
x_1 &= i\left[A_1 e^{i\frac{T_0}{q}} + (1-\delta_{q,1})A_2 e^{2i\frac{T_0}{q}} + (1-\delta_{q,1})A_3 e^{3i\frac{T_0}{q}}\right] + \text{c.c}\,,\\
y_1 &= \frac{1}{q}\left[A_1 e^{i\frac{T_0}{q}} + 2(1-\delta_{q,1})A_2 e^{2i\frac{T_0}{q}} + 3(1-\delta_{q,1})A_3 e^{3i\frac{T_0}{q}}\right] + \text{c.c}\,,\\
z_1 &= 0\,.
\end{aligned}
\tag{47}
$$

Here q is an integer and $\delta_{i,j}$ is the Kronecker symbol. Naturally, for $q = 1$, we recover the previous results by Belhaq and Houssni [1995] concerning the first period-doubling bifurcation.

Following the same procedure as above, we obtain the new set of conditions, [similar to Eqs. (15)–(19)], which vanish the secular terms in the successive approximations as

$$
\begin{aligned}
D_1 A_1 &= 0\,,\\
(1-\delta_{q,1})D_1 A_j &= 0, \quad (j = 2,\, 3)\,,
\end{aligned}
\tag{48}
$$

$$
\begin{aligned}
&\frac{2}{q}D_2A_1 - \frac{2}{q}\mu_2 A_1 + \frac{1}{q^3}\left[\left(2 - \frac{1}{1+\frac{2i}{q}}\right)\overline{A_1}A_1^2 + (1-\delta_{q,1})^2\left(8 + \frac{4}{1-\frac{i}{q}} - \frac{4}{1+\frac{3i}{q}}\right)\overline{A_2}A_2A_1\right.\\
&\quad + (1-\delta_{q,1})\left(\frac{1}{1-\frac{2i}{q}} - \frac{6}{1+\frac{2i}{q}}\right)\overline{A_1}^2A_3 + (1-\delta_{q,1})^2\left(18 + \frac{6}{1-\frac{2i}{q}} - \frac{6}{1+\frac{4i}{q}}\right)\overline{A_3}A_3A_1\\
&\quad \left. + (1-\delta_{q,1})^3\left(\frac{3}{1-\frac{i}{q}} - \frac{4}{1+\frac{4i}{q}}\right)\overline{A_3}A_2^2\right] = 0\,,
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
&\frac{4(1-\delta_{q,1})}{q}D_2A_2 - \frac{4(1-\delta_{q,1})}{q}\mu_2 A_2 + \frac{2}{q^3}\left[(1-\delta_{q,1})^3\left(8 - \frac{4}{1+\frac{4i}{q}}\right)\overline{A_2}A_2^2\right.\\
&\quad + (1-\delta_{q,1})\left(2 + \frac{4}{1+\frac{i}{q}} - \frac{4}{1+\frac{3i}{q}}\right)\overline{A_1}A_1A_2 + (1-\delta_{q,1})^3\left(18 + \frac{3}{1-\frac{i}{q}} - \frac{12}{1+\frac{5i}{q}}\right)\overline{A_3}A_3A_2\\
&\quad \left. + (1-\delta_{q,1})^2\left(\frac{3}{1+\frac{i}{q}} - \frac{6}{1+\frac{4i}{q}} + \frac{4}{1-\frac{i}{q}}\right)\overline{A_2}A_1A_3\right] = 0\,,
\end{aligned}
\tag{50}
$$

$$\frac{6(1-\delta_{q,1})}{q}D_2A_3 - \frac{6(1-\delta_{q,1})}{q}\mu_2 A_3 + \frac{3}{q^3}\left[\frac{(1-\delta_{q,1})^3}{1+\frac{2i}{q}}A_1^3 + (1-\delta_{q,1})\left(2+\frac{6}{1+\frac{2i}{q}}-\frac{6}{1+\frac{4i}{q}}\right)\overline{A_1}A_1A_3\right.$$

$$+(1-\delta_{q,1})^3\left(18-\frac{9}{1+\frac{6i}{q}}\right)\overline{A_3}A_3^2 + (1-\delta_{q,1})^2\left(4+\frac{3}{1+\frac{i}{q}}-\frac{12}{1+\frac{5i}{q}}\right)\overline{A_2}A_2A_3$$

$$\left.+(1-\delta_{q,1})^2\left(\frac{4}{1+\frac{i}{q}}-\frac{4}{1+\frac{4i}{q}}\right)\overline{A_1}A_2^2\right] = 0\,. \tag{51}$$

Substituting $A_j(T_1, T_2) = \frac{1}{2}a_j(T_1, T_2)e^{i\beta_j(T_1,T_2)}$ (where a and β are real quantities) into Eqs. (49)–(51), separating real and imaginary parts, we obtain, for $q = 1$, the same amplitude equation as in [Belhaq & Houssni, 1995],

$$\frac{da_1}{dT_2} = \mu_2 a_1 - \frac{9}{40}a_1^3\,, \tag{52}$$

and after integration of (52), we recover the value of a given by Eq. (6).

For $q = 2$, the set of Eqs. (48)–(51) reduces to the following system

$$\frac{da_1}{dT_2} = \mu_2 a_1 - \frac{3}{64}a_1^3 - \frac{81}{260}a_2^2 a_1 - \frac{99}{160}a_3^2 a_1 - \frac{1}{20}a_2^2 a_3 + \frac{5}{64}a_1^2 a_3\,,$$

$$\frac{da_2}{dT_2} = \mu_2 a_2 - \frac{9}{40}a_2^3 - \frac{129}{1040}a_1^2 a_2 - \frac{11}{80}a_1 a_2 a_3 - \frac{1359}{2320}a_3^2 a_2\,,$$

$$\frac{da_3}{dT_2} = \mu_2 a_3 - \frac{171}{320}a_3^3 - \frac{3}{40}a_2^2 a_1 - \frac{19}{160}a_1^2 a_3 - \frac{172}{1160}a_2^2 a_3 - \frac{1}{64}a_1^3\,. \tag{53}$$

Solving Eqs. (53) in terms of stationary solutions, we obtain, after some algebraic manipulations, an approximation of the three amplitudes

$$a_1 = \frac{10^3}{3}\sqrt{\frac{26}{2126847}\mu}\,,$$

$$a_2 = \frac{63}{50}a_1\,,$$

$$a_3 = \frac{14}{25}a_1\,.$$

The stability analysis (see [Rand, 1989; Belhaq & Houssni, 1995]) leads now to the two equations simultaneously

$$-1 + \frac{20}{9}\mu_{\rm PD1} + \frac{25}{324}\mu_{\rm PD1}^3 = 0\,, \tag{54}$$

$$-1 + \frac{4714450}{2392703}\mu_{\rm PD2} + \frac{5136947}{25047448}\mu_{\rm PD2}^3 = 0\,. \tag{55}$$

Equations (54) and (55) correspond to the cases $q = 1$ and $q = 2$, respectively. Solving these equations leads to an approximation of the first and the second period-doubling bifurcations $\mu_{\rm PD1} = 0.446$ and $\mu_{\rm PD2} = 0.486$, simultaneously. To compare these critical parameter values with numerical calculations, see Sec. 5 of the present work. The

Critical parameter values	$\mu_{\rm SB}$	$\mu_{\rm PD1}$	$\mu_{\rm PD2}$
First-order approach: Rand (1989)	–	0.45	–
Numerical calculation: Nayfeh *et al.* (1990)	0.30	0.44	–
Higher-order approach: Belhaq *et al.* (1995)	0.31	0.446	–
Higher-order approach: this work	–	0.446	0.486
Numerical result: Section 5 below	0.315	0.4403	0.476

table above gives a summary on the comparison of the various approaches.

5. Numerical Study

In this section we briefly describe the dynamical behavior found numerically in system (1) in the vicinity of the two first period-doubling bifurcations. To do this, we have used the software continuation code AUTO94 [Doedel *et al.*, 1996].

The stability analysis of the Hopf bifurcation (see, for instance [Freire *et al.*, 1989]) reveals that it is supercritical: a stable symmetric periodic orbit emerges for $\mu > 0$. The evolution of this periodic orbit is schematized in the bifurcation diagram of Fig. 2. In this qualitative figure we have indicated the Hopf bifurcation by an empty square, the symmetry-breaking bifurcation by an inverted triangle and the period-doubling bifurcation by a filled circle. Note that we represent the period of the T, $2T$ and $4T$-orbits divided by 1, 2 and 4, respectively.

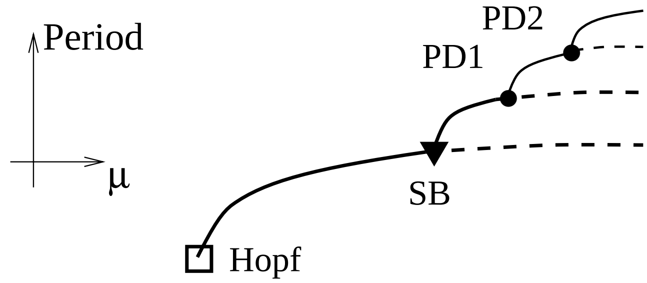


Fig. 2. Partial bifurcation diagram of the periodic orbit emerged from the Hopf bifurcation. In this qualitative figure the solid line means stable periodic orbit and the dashed line saddle periodic orbit.

First, the periodic orbit exhibits a symmetry-breaking bifurcation, `SB` ($\mu = 0.3150232$), to become a saddle orbit and a pair of asymmetric stable periodic orbits emerges.

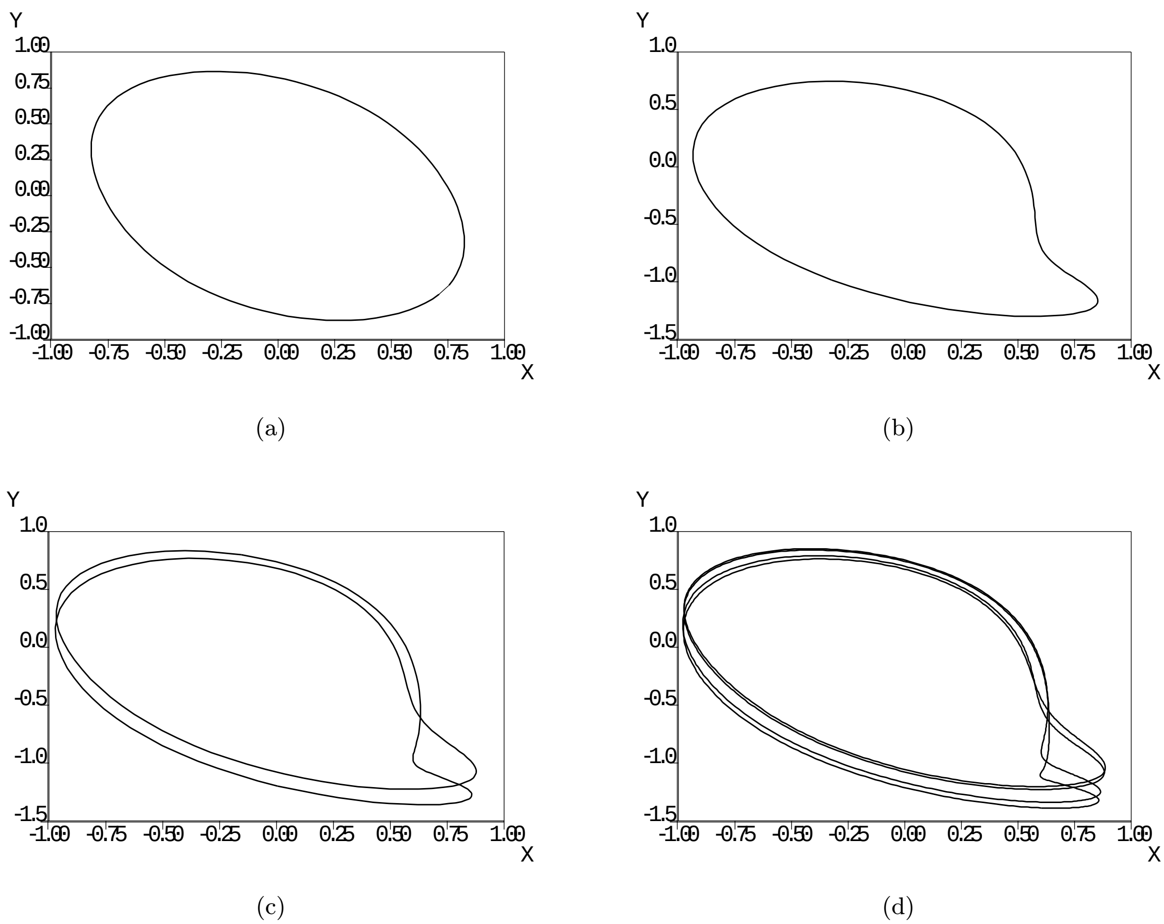


Fig. 3. (a) Symmetric periodic orbit at `SB` ($\mu = 0.3150232$). (b) Asymmetric periodic orbit, that emerged from `SB`, just at the flip bifurcation `PD1` ($\mu = 0.4403559$). (c) $2T$-orbit at the point `PD2` ($\mu = 0.4765392$). (d) A stable $4T$-orbit for $\mu = 0.485$.

Now we focus on the pair of asymmetric stable periodic orbits emerged at `SB`. These orbits become saddle when they exhibit a period-doubling bifurcation, `PD1` ($\mu = 0.4403559$). Note that this flip bifurcation was analytically predicted to occur for $\mu_{\text{PD1}} = 0.446$ (see [Belhaq & Houssni, 1995]). From such a bifurcation point a stable periodic orbit of approximately twice the period ($2T$-orbit) of the original orbit emerges.

The asymmetric $2T$-orbit (in fact a pair, due to the symmetry $(x, y, z) \to (-x, -y, z)$ the system has) born at `PD1` becomes nonstable in a flip bifurcation `PD2` ($\mu = 0.4765392$) where a $4T$-orbit emerges. The analysis developed in this work predicted this period-doubling bifurcation to occur for $\mu_{\text{PD2}} = 0.486$.

Finally, we show in Fig. 3(a) the symmetric periodic orbit at the point `SB` where it exhibits the symmetry-breaking bifurcation. The asymmetric periodic orbit (that emerged from `SB`) is drawn in Fig. 3(b) just at the point where it exhibits the flip bifurcation `PD1`. In Fig. 3(c) we have represented the $2T$-orbit at the point `PD2`. A stable $4T$-orbit is shown in Fig. 3(d), for $\mu = 0.485$.

6. Conclusions

In this paper we have investigated analytically the two first period-doubling bifurcations in a three-dimensional differential system (period-one to period-two and period-two to period-four). These two first instabilities generally organize the dynamics of such systems toward chaos. The prediction of these period-doubling is obtained by performing a stability analysis in terms of Floquet theory of the periodic orbit approximated by applying the multiple scales technique to a higher order.

By introducing suitable subharmonic terms in the first-order solution of the multiple scales procedure, we have derived two conditions allowing us to obtain the two critical values simultaneously. The subharmonic perturbation introduced to the periodic solution at the first step of multiple scales technique has given account of the dynamics of the periodic solution in the vicinity of the period-doubling branch.

The stability analysis carried out in this work for the three-dimensional systems is an adaption of the analysis developed in [Szemplinska-Stupnicka, 1990] for the one-degree-of-freedom nonlinear oscillators driven by a periodic forcing.

This adaption to a three-dimensional system gives a good enough approximation of the critical parameter values by comparison with numerical simulations.

The approach performed here to predict the period-doubling bifurcations has considered the z-direction since in our specific system the z-direction governs the stability of the periodic solution in terms of period-doubling. Note that numerical investigations showed that the bifurcation branch following Hopf bifurcation undergoes a symmetry-breaking, two first period-doublings and homoclinic bifurcations. The results of the present work and of the one focusing on homoclinic bifurcation (see [Belhaq *et al.*, 1999]) provide analytical prediction of the complete bifurcation branch.

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