ANALYTICS OF HOMOCLINIC BIFURCATIONS IN THREE-DIMENSIONAL SYSTEMS

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An analytical approach to determine critical parameter values of homoclinic bifurcations in three-dimensional systems is reported. The homoclinic orbit is supposed to be a limit of a unique periodic orbit. Hence, the multiple scales perturbation method is performed to construct an approximation of the periodic solution and its frequency. Then, two simple criteria are used. The first criterion is based on the collision between the periodic and the hyperbolic fixed point involved in the bifurcation. The second uses the infinity condition of the period of the periodic orbit. For illustration a specific system is investigated.

Keywords: Periodic orbit; homoclinic bifurcation; three-dimensional systems; multiple scales method.

1. Introduction

The dynamics of three-dimensional systems is very rich, in terms of bifurcations and new phenomena. Important new behaviors include sequences of period doubling, existence of strange attractors, bifurcations creating the horseshoe (see e.g. [Wiggins, 1988]).

On the basis of the work by Shil'nikov [1965, 1968], the chaotic behavior is exhibited in the neighborhoods of parameter space where certain homoclinic orbits occur. Thus, the location of homoclinic orbits is important in pinpointing a region in parameter space in which we may expect chaos to occur (see [Gaspard & Nicolis, 1983; Glendinning & Sparrow, 1984; Arneodo et al., 1985]).

Indeed, homoclinc orbits are of great importance from theoretical and applied points of view

[Shil'nikov, 2000]. For instance they form the profiles of traveling wave solutions in reaction–diffusion problems (see [Fitzhugh, 1961]). In static-dynamics analogies, a homoclinic orbit corresponds to a spatially localized post-buckling state [Thompson & Van der Heijden, 1999]. For more applications see [Sparrow, 1982; Nekorkin & Kazantsev, 1996].

In fact, the investigation of homoclinic bifurcations in three-dimensional systems has received more attention from the numerical point of view (see [Algaba et al., 1998; Khibnik et al., 1993]). However, Glendinning and Sparrow [1984] studied the local behavior near homoclinic orbits to stationary points of saddle-focus type. They described how a periodic orbit approaches homoclinicity using a Poincaré map composed of two maps; one given by the linear flow near the saddle-focus and the other by rigid motion along the homoclinic orbit outside

its neighborhood. The same method was used by Gaspard *et al.* [1984] to study the local behavior near homoclinic orbits. Morozov [1998] investigated three-dimensional systems which are close to two-dimensional Hamiltonian systems. He averaged over the fast variable to reduce the problem to a planar one. On the other hand, Mikhlin [2000] used a quasi-Padé approximant method to construct homoclinic orbits of the Lorenz equation.

In the present paper we are interested to perform analytically a criterion for predicting bifurcation of homoclinic orbits emanating from periodic orbits. In other words, we focus on simple (principal) homoclinic orbits.

The criterion of homoclinicity, proposed here, is based on the collision between the periodic orbit and the fixed point involved in the bifurcation. This criterion developed initially [Belhaq, 1998; Belhaq *et al.*, 2000a] to predict homoclinic bifurcations in planar autonomous systems is adapted here to derive a critical value of the homoclinic bifurcation in three-dimensional systems. Mathematically speaking, it was shown, for the planar case, in [Belhaq *et al.*, 2000a] that the Melnikov condition and the proposed method are equivalent. Note however, that the proposed criterion is accessible via approximations of periodic orbits. The Melnikov approach, on the other hand, circumvents periodic orbits by aiming directly at the separatrices. It is worth pointing out that the proposed criterion depends on the method used to compute an approximation of the periodic orbit.

The present work can be viewed as a generalization of the results given in [Lakrad *et al.*, 2000; Belhaq *et al.*, 2000a, 2000b], where a similar criterion was used to study prototypic examples. In fact, we will focus our attention to the application of the criterion of homoclinicity to generic three-dimensional autonomous ordinary differential equations i.e. without symmetry.

Hence, in Sec. 2, we present the criterion of homoclinicity and discuss the domain of its validity. For illustration, in Sec. 3 we include explicit calculations for a specific system. Comparisons to numerical simulations are also provided. A summary is given in Sec. 4.

2. Formulation of the Method

We consider a three-dimensional autonomous ordinary differential equation, with an analytic vector field (in fact only C^2 is required, see [Wiggins, 1988]) whose linearized part has a hyperbolic fixed point S (which could be, without loss of generality, the origin i.e. $S(0, 0, 0)$). We assume here that

(H1) At a critical parameter value $\mu_c = 0$ the system possesses a nondegenerate homoclinic orbit Γ, connecting the hyperbolic fixed point S to itself.

(H2) The homoclinic orbit Γ is a limit of a unique periodic orbit γ which is called the *principal periodic orbit*.

(H3) The system is generic i.e. without symmetry.

(H4) The system has another equilibrium B.

It should be clear that the three eigenvalues of the linearized vector field at the hyperbolic fixed point S can be of two possible types:

- (a) *Saddle* λ_1, λ_2, λ_3; λ_i real, λ_1, $\lambda_2 < 0$, $\lambda_3 > 0$
- (b) *Saddle-focus* $\rho \pm i\omega$; $\rho < 0$, $\lambda > 0$.

All other cases of hyperbolic fixed points may be obtained from (a) and (b) by time reversal [Wiggins, 1988]. In fact, the nature of the orbits structure near the homoclinic orbit depends considerably on the nature of the eigenvalues of the linearized vector field at S and the existence of symmetries (for more details see for instance [Wiggins, 1988; Khibnik *et al.*, 1993]). In the case (a) the ODE can be written as

$$
\begin{aligned}
\dot{x} &= \lambda_1 x + f_1(x, y, z; \mu) \\
\dot{y} &= \lambda_2 y + f_2(x, y, z; \mu) \qquad (x, y, z, \mu) \in \mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^1 \\
\dot{z} &= \lambda_3 z + f_3(x, y, z; \mu)
\end{aligned}
\tag{1}
$$

here f_i are analytic functions of their arguments. They vanish with their first derivatives for $\mu = 0$ in S.

In the case (b) the ODE can be written as

$$
\begin{aligned}
\dot{x} &= \rho x - \omega y + f_1(x, y, z; \mu) \\
\dot{y} &= \omega x + \rho y + f_2(x, y, z; \mu) \\
\dot{z} &= \lambda z + f_3(x, y, z; \mu)
\end{aligned}
\tag{2}
$$

The f_i have the same characteristics as in the case (a).

For the case (a) the assumption (H1) is related mainly to the saddle value which must be nonzero i.e. $\lambda_3 + \max(\lambda_1, \lambda_2) \neq 0$. In the case of saddle-focus the assumption (H1) is related to the saddle index which must be greater than one i.e. $-\rho/\lambda > 1$. For more details see [Shilnikov *et al.*, 2001].

It was shown (see [Glendinning & Sparrow, 1984; Wiggins, 1988; Shilnikov *et al.*, 2001]), under the nondegeneracy assumption (H1) and the genericity assumption (H3) that one periodic orbit bifurcates from the homoclinic loop. Hence, the assumption (H2) can be realized under these latter conditions. In fact, it is expected that the fixed point B undergoes a Hopf bifurcation when μ is varied. If we go beyond the Hopf bifurcation, the instability of B increases and the formed limit cycle γ grows till it runs into the stable manifold of S and thus forms the principal homoclinic orbit.

The strategy proposed in this paper to locate the principal homoclinic bifurcation consists of two steps

(1) Construction of an approximation of the periodic orbit γ and its period.
(2) The collision between γ and S.

In fact, standard methods [Nayfeh & Mook, 1979] can be used to construct the periodic solution. In our application, we have used the multiple scales method. Indeed, the choice of the method is crucial in having good results. It can induce new restrictions like the smallness of some quantities in the case of perturbation methods.

The criterion proposed in (2) is based on vanishing the distance between the periodic solution γ and the hyperbolic fixed point S. To be more specific, let $\mathbf{x}(\mu, t) = (x(\mu, t), y(\mu, t), z(\mu, t))$ be an approximation of the periodic orbit γ. The criterion of collision can be stated mathematically as follows

$$\lim_{\mu \to \mu_c} \{S\} \cap \{\mathbf{x}(\mu, t)\} = \{S\} \qquad \text{for a given } t. \quad (3)$$

Since the homoclinic orbit is structurally unstable, then the critical parameter μ_c is unique.

In fact, the vanishing of the distance between the periodic orbit γ and the stable manifold of S (which is assumed to be two-dimensional) is sufficient for having a homoclinic orbit.

On the other hand, the homoclinic orbit Γ can be viewed as a periodic orbit of period infinity. Indeed, some standard methods used to approximate periodic solutions give also an approximation of the period. Hence, a criterion of homoclinicity can be obtained by tending the approximated period to infinity or by imposing the nullity of the frequency.

3. Application

Consider the following three-dimensional system

$$\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= z, \\
\dot{z} &= -z - \mu_1 y + \mu_2 x - x^2,
\end{aligned} \quad (4)$$

originally introduced by Arneodo *et al.* [1985] in their discussion of the simplest systems of the normal form which exhibit the generic features of multiple periodicities and chaos as period-one orbits approach homoclinicity. Here μ_1 and μ_2 are control parameters. The dot denotes the time derivative. The system (4) has two stationary points: the origin, $S(x = y = z = 0)$ and $B(x = \mu_2, x = z = 0)$. We shall work in the region of parameter space where $\mu_1, \mu_2 > 0$. Hence, the origin is always unstable, its stability analysis reveals that it is a saddle for

$$\frac{2}{27} - \frac{\mu_1}{3} - \sqrt{-\frac{4}{27}\left(\mu_1 - \frac{1}{3}\right)^3} \leq \mu_2 \leq \frac{2}{27} - \frac{\mu_1}{3}$$
$$+ \sqrt{-\frac{4}{27}\left(\mu_1 - \frac{1}{3}\right)^3}. \quad (5)$$

On the other hand, the origin has a complex conjugate pair of eigenvalues with negative real part and one positive real eigenvalue outside the region (5). Hence, in the region given by (5) we will have a saddle loop, while outside there will be a saddle-focus loop.

The stability analysis of the linearized equations around B indicates that this fixed point becomes unstable at $\mu_2 = \mu_1$ through a Hopf bifurcation which creates a stable periodic orbit. This periodic orbit is called the *principal periodic orbit* since it forms the main homoclinic orbit.

Equation (4) has been studied numerically in [Glendinning & Sparrow, 1984; Arneodo *et al.*, 1985], to locate local and global behaviors near homoclinic orbits. On the other hand, Phillipson and Schuster [1998] used an asymptotic averaging formalism to analytically construct periodic solutions

of Eq. (4) and to predict the period doubling and reverse period doubling bifurcations.

3.1. *Approximation of the periodic orbit*

To construct the periodic orbit we use the multiple scales method (MSM) [Nayfeh & Mook, 1979]. Set

$$X = x - \frac{\mu_1 + \mu_2}{2} \tag{6}$$

$$H = X + \dot{X}, \tag{7}$$

then Eq. (4) becomes

$$\ddot{H} + \mu_1 H = \varepsilon(h - X^2), \tag{8}$$

where ε is a small positive dimensionless parameter, artificially introduced to serve as a bookkeeping device in obtaining the approximate solution [Nayfeh & Mook, 1979]; and $h = (\mu_2^2 - \mu_1^2)/4$. In fact, the approach used here to solve Eq. (4) imposes that h be small. Hence, a good approximation of the periodic solution and a satisfactory condition of homoclinicity should be expected for small h.

It is worth noting that MSM is characterized by the introduction of independent scales of time and consequently the transformation of the ordinary differential equations to a set of partial differential equations.

Hence, the solution of Eq. (8) and the scales of time are expressed in terms of ε as follows

$$H(t; \varepsilon, N) = \sum_{i=0}^{N} \varepsilon^i H_i(T_0, T_1, \ldots, T_N) + \mathcal{O}(\varepsilon^{N+1}), \tag{9}$$

$$T_i = \varepsilon^i t. \tag{10}$$

Here $\mathcal{O}$ is the order symbol which measures the relative order of magnitude of various quantities, and $T_i(i = 0, \ldots, N)$ are considered independent scales of time which get longer as the integer i increases. Thus, T_0 is a fast time scale on which the main oscillatory behavior occurs and T_i (where $i > 0$) are slow time scales characterizing modulations of amplitudes and phases. Using the chain rule we have

$$\frac{d}{dt} = \sum_{i=0}^{N} \varepsilon^i D_i \tag{11}$$

where $D_i = \partial/\partial T_i$. Thus, using the MSM to Eq. (8) we obtain at different orders of ε

- order $\mathcal{O}(\varepsilon^0)$

$$D_0^2 H_0 + \mu_1 H_0 = 0, \tag{12}$$

which has the following solution

$$H_0 = A \exp(i\sqrt{\mu_1}\, T_0) + \text{c.c.}, \tag{13}$$

where c.c. denotes the conjugate of the required terms. Consequently, from Eq. (7)

$$X_0 = N A \exp(i\sqrt{\mu_1}\, T_0) + \text{c.c.}, \tag{14}$$

where $N = (1 - i\sqrt{\mu_1})/(1 + \mu_1)$.

- order $\mathcal{O}(\varepsilon^1)$

$$D_0^2 H_1 + \mu_1 H_1 = -2D_0 D_1 H_0 + h - X_0^2. \tag{15}$$

The condition of elimination of secular terms from Eq. (15) leads to

$$D_1 A = 0. \tag{16}$$

Hence, the particular solution of Eq. (15) is given by

$$H_1 = \frac{N^2}{3\mu_1} A^2 \exp(i2\sqrt{\mu_1}\, T_0) - \frac{N\bar{N}}{\mu_1} A\bar{A}$$

$$+ \frac{h}{2\mu_1} + \text{c.c.} \tag{17}$$
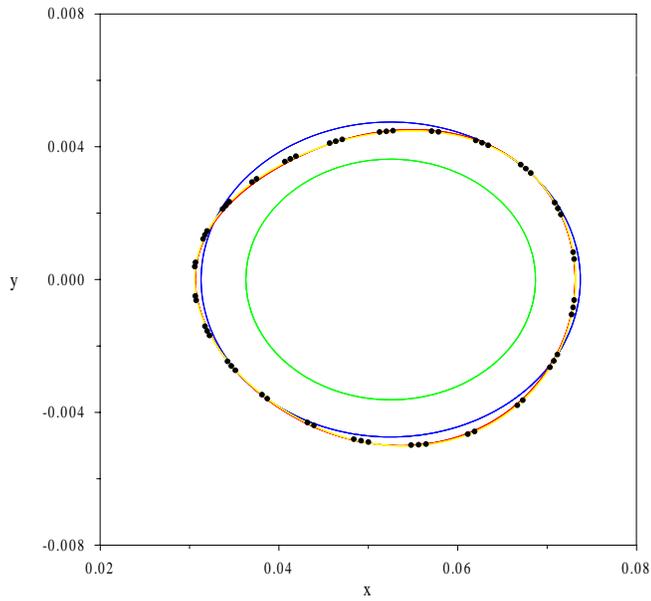
From Eq. (7) we have

$$X_1 = \frac{N^2}{3\mu_1(1 + i2\sqrt{\mu_1})} A^2 \exp(i2\sqrt{\mu_1}\, T_0)$$

$$- \frac{N\bar{N}}{\mu_1} A\bar{A} + \frac{h}{2\mu_1} + \text{c.c.} \tag{18}$$

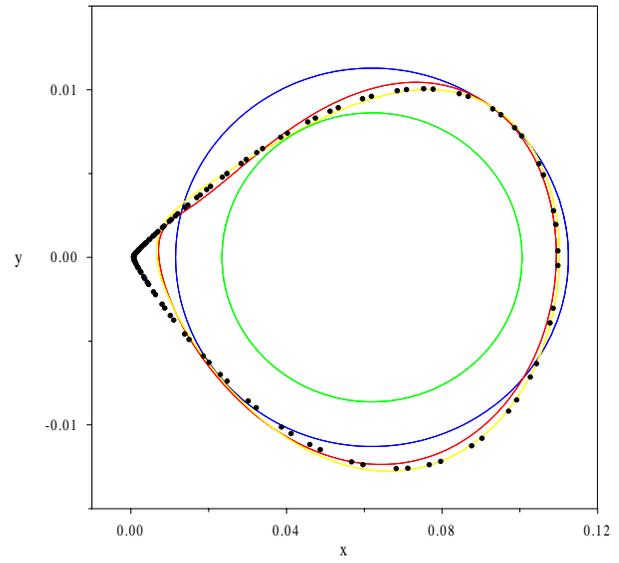- order $\mathcal{O}(\varepsilon^2)$

$$D_0^2 H_2 + \mu_1 H_2 = -2D_0 D_2 H_0 - 2D_0 D_1 H_1$$

$$- D_1^2 H_0 - 2X_0 X_1. \tag{19}$$

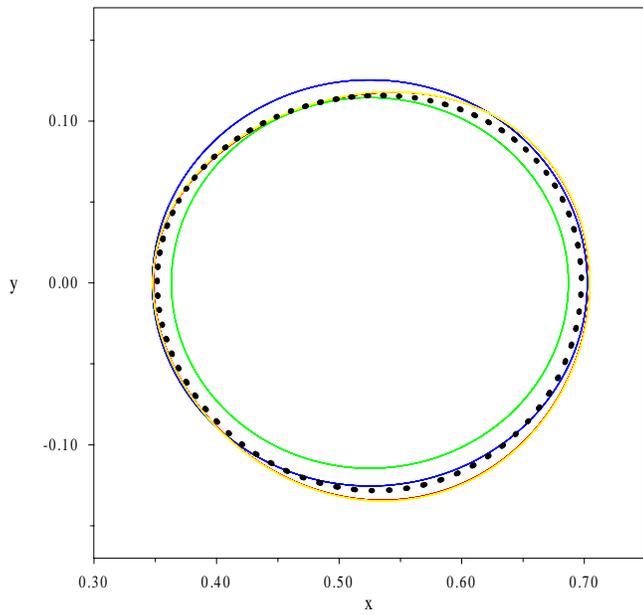The elimination of secular terms leads to the following equation

$$i\sqrt{\mu_1}(D_2 A) = -\frac{Nh}{\mu_1} A + \frac{2N^2\bar{N}}{\mu_1} A^2\bar{A}$$

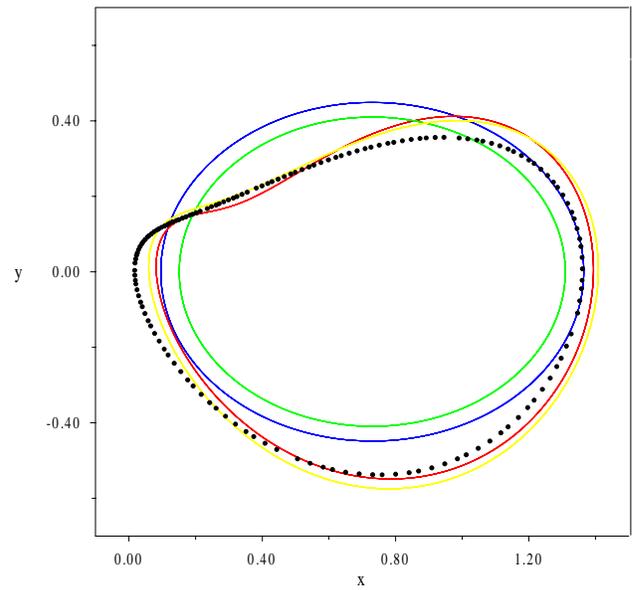$$- \frac{N^2\bar{N}}{3\mu_1(1 + i2\sqrt{\mu_1})} A^2\bar{A}. \tag{20}$$

Fig. 1. Comparisons of different approximations of periodic orbits of Eq. (4) in the plane (x, y). The dotted curve is the computer generated solution. The green curve, is the approximate analytic solution given by Phillipson and Schuster [1998]. The blue, red and yellow curves are respectively the orders zero, one and two of the solution given by MSM in Eq. (25). For the plots (a) and (b) $\mu_1 = 0.05$ and μ_2 is 0.055 and 0.074, respectively. These two values correspond to the case where S is a saddle. For the plots (c) and (d) $\mu_1 = 0.5$ and μ_2 is 0.55 and 0.96, respectively. These two values correspond to the case where S is a saddle-focus. The plots (a) and (c) show the periodic solution just after the Hopf bifurcation of B, while plots (b) and (d) show it near the principal homoclinic orbit.

Set $A = (a/2) \exp(i\theta)$ and substitute it in Eq. (20), after separation of real and imaginary parts, we obtain the following modulation equations of amplitude and phase

$$\dot{a} = \frac{h}{\mu_1(1+\mu_1)}a - \frac{1+8\mu_1}{4\mu_1(1+4\mu_1)(1+\mu_1)^2}a^3, \quad (21)$$

$$\dot{\theta} = \frac{h}{\mu_1\sqrt{\mu_1}(1+\mu_1)} - \frac{a^2(5+26\mu_1)}{12\mu_1\sqrt{\mu_1}(1+4\mu_1)(1+\mu_1)^2}. \quad (22)$$

The amplitude and the frequency of the periodic solution are, respectively given, as follows (see [Nayfeh & Mook, 1979])

$$a_s^2 = \frac{4h(1+\mu_1)(1+4\mu_1)}{1+8\mu_1} \quad (23)$$

$$\Psi = \sqrt{\mu_1} + \frac{h}{\mu_1^{3/2}(1+\mu_1)}$$
$$- \frac{a_s^2(5+26\mu_1)}{12\mu_1^{3/2}(1+4\mu_1)(1+\mu_1)^2}. \quad (24)$$

Here, the amplitude a_s is obtained by solving the algebraic equation $\dot{a} = 0$.

The solution of Eq. (4) up to the second order of ε is given by

$$x(t) = \frac{\mu_1 + \mu_2}{2} + \frac{a_s}{(1+\mu_1)}\cos(\Psi t)$$
$$+ \frac{a_s\sqrt{\mu_1}}{(1+\mu_1)}\sin(\Psi t) + R_1 a_s^2 \cos(2\Psi t)$$
$$+ R_2 a_s^2 \sin(2\Psi t) - \frac{a_s^2}{2\mu_1(1+\mu_1)} + \frac{h}{\mu_1}$$
$$+ R_3 a_s^3 \cos(3\Psi t) + R_4 a_s^3 \sin(3\Psi t)$$
$$+ (R_5 a_s + R_6 a_s^3)\cos(\Psi t)$$
$$+ (R_7 a_s + R_8 a_s^3)\sin(\Psi t) \quad (25)$$

where,

$$R_1 = \frac{1-5\mu_1}{6\mu_1(1+\mu_1)^2(1+4\mu_1)} \quad (26)$$

$$R_2 = \frac{\sqrt{\mu_1}(2+\mu_1)}{3\mu_1(1+\mu_1)^2(1+4\mu_1)} \quad (27)$$

$$R_3 = \frac{23\mu_1^2 - 24\mu_1 + 1}{48\mu_1^2(1+\mu_1)^3(1+4\mu_1)(1+9\mu_1)} \quad (28)$$

$$R_4 = \frac{8 - 34\mu_1 + 6\mu_1^2}{48\mu_1^{3/2}(1+\mu_1)^3(1+4\mu_1)(1+9\mu_1)} \quad (29)$$

$$R_5 = \frac{h(\mu_1 - 3)}{\mu_1(1+\mu_1)^3} \quad (30)$$

$$R_6 = \frac{3-\mu_1}{2\mu_1(1+\mu_1)^4} - \frac{5-7\mu_1}{12\mu_1(1+4\mu_1)(1+\mu_1)^4} \quad (31)$$

$$R_7 = \frac{h(1-3\mu_1)}{\mu_1\sqrt{\mu_1}(1+\mu_1)^3} \quad (32)$$

$$R_8 = \frac{3\mu_1 - 1}{2\mu_1\sqrt{\mu_1}(1+\mu_1)^4}$$
$$+ \frac{1 - 9\mu_1 + 2\mu_1^2}{12\mu_1\sqrt{\mu_1}(1+4\mu_1)(1+\mu_1)^4} \quad (33)$$

In Fig. 1, we show for different values of μ_1 and μ_2, the comparisons between the results of MSM at different orders, the analytical solution given by Phillipson and Schuster [1998] and numerical simulations in the plane (x, y). It is worth noting, that MSM gives the best analytical approximation, specially, for small h. On the other hand, the approximation of the periodic solution is improved for higher order MSM even near homoclinic loops. In fact, for $\mu_1 = 0.05$ the origin is a saddle while for $\mu_1 = 0.5$ it is a saddle-focus.

3.2. *Homoclinicity conditions*

For Eq. (4) the main role in the application of the collision criterion (3) is applied by the component $x(t)$ of the solution. It is easy to show that the components $y(t)$ and $z(t)$ oscillate around zero for all $\mu_1 < \mu_2 < \mu_{2c}$, i.e. they touch the hyperbolic fixed point S in these two directions.

Hence, the order $\mathcal{O}(\varepsilon^0)$ approximation of the solution $x(t)$, given in Eq. (25), leads to the following analytical homoclinicity condition

$$\mu_{2c} = \mu_1 \frac{5+24\mu_1}{3+8\mu_1}. \quad (34)$$

For higher order approximations of $x(t)$ only numerical conditions can be obtained.

On the other hand, frequency nullity criterion applied to the approximate frequency Ψ given in Eq. (24) leads to the following homoclinicity condition

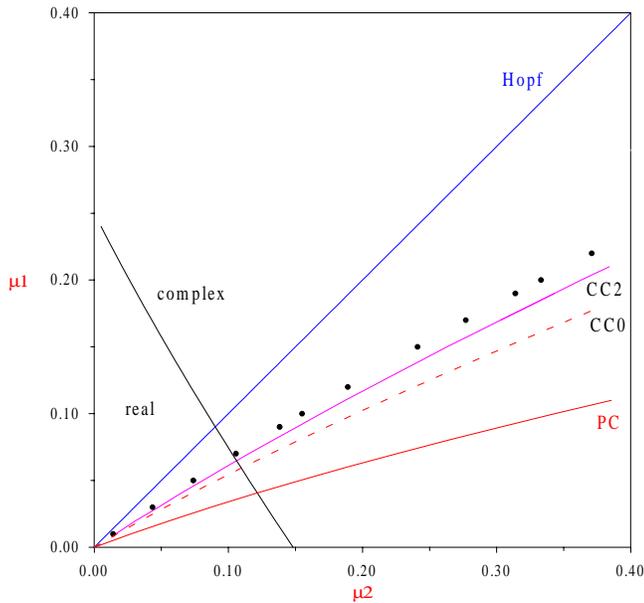$$\mu_2 = \mu_1\sqrt{7+48\mu_1}. \quad (35)$$

Fig. 2. The (μ_2, μ_1) parameter space of Eq. (4) shows the eigenvalues at the origin, the locus of the Hopf bifurcation from B and the principal homoclinic bifurcation criteria. The dotted curve corresponds to the numerical values, (PC) corresponds to the period criterion given in Eq. (35), (CC0) is the collision criterion given in Eq. (34) and (CC2) is the collision criterion using the second order of $x(t)$ given in Eq. (25).

In Fig. 2, we show comparisons between the different criteria of homoclinicity and numerical results. The analytical criteria are better for small μ_1 and μ_2, in fact, for small h as it was imposed by the use of MSM. It is worth noting, that the collision criterion gives better results than the nullity of the frequency for the leading order of approximation. The same conclusion was derived for planar autonomous systems [Belhaq *et al.*, 1999].

4. Conclusion

An analytical approach to determine critical parameter values of homoclinic bifurcations in generic three-dimensional systems is reported. The homoclinic orbit is supposed to be a limit of a unique periodic orbit. This latter was approximated by the multiple scales perturbation method which gives also an approximation of the frequency. In fact, the method gives good results even near the principal homoclinic orbit when the smallness of some parameters is respected. Then, two simple criteria are used. The first criterion is based on the vanishing of the distance between the periodic orbit and the hyperbolic fixed point involved in the bifurcation. The second uses the infinity condition of

the period of the homoclinic orbit. The accuracy of these criteria depends mainly on the accuracy of the approximated periodic solution. Comparisons of these criteria to numerical simulations and other works are provided for a specific three-dimensional system.

However, the use of the proposed approach to the case of three-dimensional systems with symmetries and to subsidiary homoclinic orbits waits to be explored.

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