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Asymptotic solutions for a damped non-linear quasi-periodic Mathieu equation

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Abstract

Quasi-periodic (QP) solutions of a weakly damped non-linear QP Mathieu equation are investigated near a double primary parametric resonance. A double multiple scales method is applied to reduce the original QP oscillator to an autonomous system performing two successive reduction. The problem for approximating QP solutions of the original system is then transformed to the study of stationary regimes of the induced autonomous system. Explicit analytical approximations to QP oscillations are obtained and comparisons to numerical integration of the original QP oscillator are provided. © 2001 Elsevier Science Ltd. All rights reserved.

1. Introduction

In this paper, we investigate analytical approximations of quasi-periodic (QP) solutions to a weakly damped non-linear QP Mathieu equation having incommensurate frequencies in the form

$$\ddot{x}+\alpha\dot{x}+(\omega^2+h\cos(\Omega t)+\rho\cos(\upsilon t))x+\xi x^3=0, \tag{1}$$

where damping α, non-linear component ξ, excitation amplitudes h and ρ and frequency v are small. The quantities ω and Ω are the proper and parametric frequencies. An overdot denotes differentiation with respect to time t. Eq. (1) can serve as a one-mode model to mechanical systems submitted to two simultaneous additional parametric modulations having incommensurate frequencies.

Yagazaki and co-workers [1] analyzed the dynamics of an oscillator subjected to parametric and external excitations in a incommensurate case and with a weakly cubic non-linear component. The Van der Pol transformation was applied to obtain a linear approximation of periodic solution of the averaged system. In a recent work, Belhaq and Houssni [2] developed a strategy for constructing an explicit asymptotic expansion of QP solutions to an oscillator with cubic and quadratic non-linearities, subjected to parametric and external excitations having incommensurate frequencies. The method consists of reducing the original QP system to the so-called second autonomous reduced system (RS) performing two successive averaging.

Zounes and Rand [3] studied the linear case ($\xi=0$) of Eq. (1) and analyzed the stability chart

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using different methods aiming on the numerical integration, the harmonic balance, a standard perturbation technique and the Lyapunov exponent. Comparison of these various methods was reported and a very good agreement was obtained. A variant of Eq. (1) was considered by Gumowski [4], in which the QP effect was introduced by modulating the amplitude of the parametric excitation to the standard Mathieu equation. The generalized averaging technique [5] was performed in a Cartesian formulation to reduce the QP Mathieu equation into a differential system with periodic coefficients. The question concerning the approximations of periodic solutions of the reduced system was pointed out but not considered. Other papers have focused on the transition curves for QP systems. Schweitzer [6] studied conditions for the stability or instability of solutions to linear QP system. Weidenhammer [7,8] applied a perturbation method to linear and non-linear QP Mathieu equations and determined analytic expression for transition curves.

Here, we focus attention on the construction of explicit approximate analytical solutions of Eq. (1) applying the multiple scales twice to carry out a double reduction leading to a second autonomous RS. Note that in [2], we have combined the multiple scales with the Bogolioubov–Mitropolsky method [5] to perform the double reduction. Comparison was carried out only to validate the approximation obtained for the first RS. The main purpose here is to examine the validity of the approximate QP solutions comparing with a direct numerical integration of the original equation (1).

We will restrict ourselves to the study of QP solutions near a double primary parametric resonance, namely the generating resonance 1 : 2 and the satellite [5] resonance 1 : 2.

The paper is organised as follows. In Section 2, we apply the multiple scales to obtain a first RS system having periodic terms. Analysis of stability and determination of response curves of the periodic solutions of the RS are reported. In Section 3, we derive the second autonomous RS by applying again the multiple scales to the first RS. Explicit analytical approximations of QP solutions are obtained and compared to numerical integration. We conclude in Section 4.

2. First reduced system

Following [1,2], we introduce two small parameters ε and μ, such that $0 < \varepsilon \ll \mu \ll 1$ and let: $\alpha = \mu\tilde{\alpha}$, $\tilde{\alpha} = \varepsilon^2\tilde{\tilde{\alpha}}$, $\rho = \varepsilon\tilde{\rho}$, $h = \mu\tilde{h}$, $\tilde{h} = \varepsilon^2\tilde{\tilde{h}}$, $\Omega = \varepsilon\tilde{\Omega}$ and $\xi = \varepsilon^2\tilde{\xi}$. The introduction of the two small parameters ε and μ in Eq. (1) allows one to carry out two successive reductions. Thus, it can be written as

$$\ddot{x} + \omega^2 x = -\varepsilon(\tilde{\rho}\cos(\upsilon t)x) - \varepsilon^2(\mu\tilde{\tilde{\alpha}}\dot{x} + \mu\tilde{\tilde{h}}\cos(\varepsilon\tilde{\Omega}t)x + \tilde{\xi}x^3), \tag{2}$$

which contains two times, a 'fast' time t and a 'slow' time $\tau = \varepsilon t$, which are assumed to be independent.

Using the multiple scales technique [9,10], an approximation of QP solutions to Eq. (2) is sought in the form

$$x(t) = x_0(T_0, T_1, T_2) + \varepsilon x_1(T_0, T_1, T_2) + \varepsilon^2 x_2(T_0, T_1, T_2) + \cdots, \tag{3}$$

where $T_n = \varepsilon^n T_0$. In terms of the variable T_n, the time derivative becomes $\mathrm{d}/\mathrm{d}t = \varepsilon D_1 + \varepsilon^2 D_2 + \cdots$, where $D_n = \partial/\partial T_n$. Substituting Eq. (3) into (2) and equating coefficients of like powers of ε, one obtains the following systems:

$$D_0^2 x_0 + \omega^2 x_0 = 0, \tag{4}$$

$$D_0^2 x_1 + \omega^2 x_1 = -2D_0 D_1 x_0 - \tilde{\rho}\cos(\upsilon T_0)x_0, \tag{5}$$

$$D_0^2 x_2 + \omega^2 x_2 = -2D_0 D_1 x_1 - 2D_0 D_2 x_0 - D_1^2 x_0 - \tilde{\alpha} D_0 x_0 - \tilde{\rho}\cos(\upsilon T_0)x_1 - \tilde{h}\cos(\tilde{\Omega}T_1)x_0 - \xi x_0^3. \tag{6}$$

The general solution of (4) can be expressed in the form

$$x_0(T_0, T_1, T_2) = A(T_1, T_2)\exp(\mathrm{i}\omega T_0) + \mathrm{cc}, \tag{7}$$

where cc denotes the complex conjugate of the preceding terms and A is to be determined by the elimination of secular terms from the next-order equations. Substituting Eq. (7) into (5) we obtain

$$D_0^2 x_1 + \omega^2 x_1 = -2\mathrm{i}\omega D_1 A e^{\mathrm{i}\omega T_0} - \tfrac{1}{2}\tilde{\rho}A e^{\mathrm{i}(\omega+\upsilon)T_0} - \tfrac{1}{2}\tilde{\rho}\bar{A}e^{\mathrm{i}(\nu-\omega)T_0} + \mathrm{cc}. \tag{8}$$

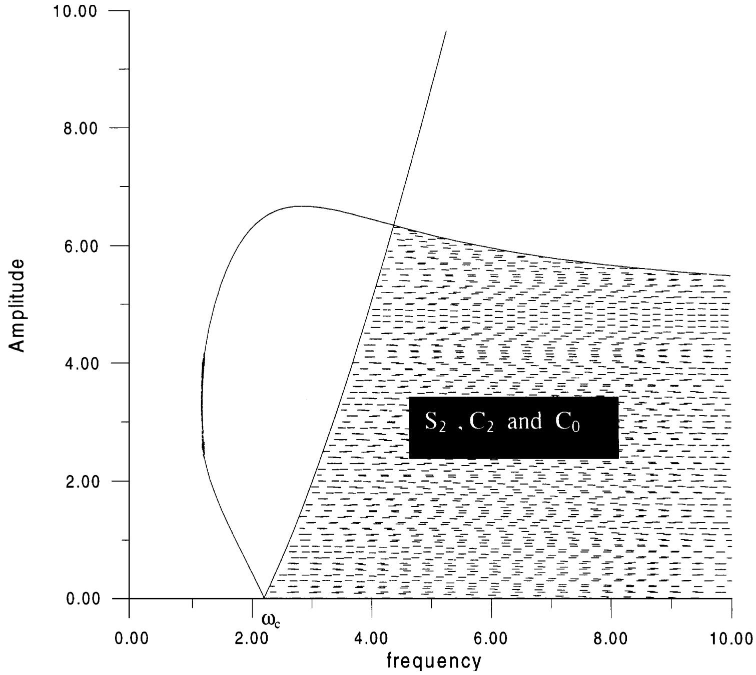

Fig. 1. Bifurcation curves of quasi-periodic solutions in the (ρ, υ) plane for $\omega = 1.1$.

We will restrict our analysis to QP motions in the vicinity of the primary resonance one-half ($\upsilon \approx 2\omega$). Introducing the detuning parameter σ according to $\upsilon = 2\omega + \varepsilon\sigma$, substituting into Eq. (8) and eliminating the secular terms, we find

$$D_1 A = \frac{\mathrm{i}}{4\omega}\tilde{\rho}\bar{A}\mathrm{e}^{\mathrm{i}\sigma T_1}, \tag{9}$$

where the overbar indicates the complex conjugate. The solution of Eq. (8) is then written as

$$x_1(T_0, T_1, T_2) = \frac{\tilde{\rho}}{2\upsilon(2\omega + \upsilon)} A(T_1, T_2)\mathrm{e}^{\mathrm{i}(\omega+\upsilon)T_0} + \mathrm{cc}. \tag{10}$$

Substituting Eqs. (7) and (10) into Eq. (6) and eliminating the secular terms, one obtains

$$-2\mathrm{i}\omega D_2 A - D_1^2 A - \mathrm{i}\omega\tilde{\alpha}A - \tilde{h}A\cos(\tilde{\Omega}T_1)$$
$$-3\xi A^2\bar{A} - \rho^2\frac{A}{4\upsilon(2\omega + \upsilon)} = 0. \tag{11}$$

Elimination of secular terms up to second order is given by

$$\frac{\mathrm{d}A}{\mathrm{d}t} = D_0 A + \varepsilon D_1 A + \varepsilon^2 D_2 A. \tag{11a}$$

Letting $A = \frac{1}{2}a\mathrm{e}^{\mathrm{i}\beta}$ where a and β are real functions, substituting into Eq. (11a), separating real and imaginary parts, we obtain the first reduced modulation equations of amplitude and phase system in the polar form

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\mu\frac{1}{2}\tilde{\alpha}a - \frac{1}{4\omega}\left(2 - \frac{\upsilon}{2\omega}\right)\rho a\sin(2\gamma),$$
$$a\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{\upsilon - 2\omega}{2}a - \frac{1}{4\omega}\left(2 - \frac{\upsilon}{2\omega}\right)\rho a\cos(2\gamma)$$
$$- \frac{1}{32\omega^3}\rho^2 a - \frac{3}{8\omega}\xi a^3 - \frac{1}{8\omega\upsilon(2\omega + \upsilon)}\rho^2 a$$
$$- \mu\frac{1}{2\omega}\tilde{h}a\cos(\Omega T_1), \tag{12}$$

where $\gamma = \frac{1}{2}\sigma T_1 - \beta$.

The periodic solutions of this system (12) correspond to the QP solutions of (1) near the generating resonance 1 : 2. Here μ which appears in system (12) is considered as a perturbation parameter.

The stationary non-trivial solutions ($a_0 \neq 0$) of the unperturbed ($\mu = 0$) equations of (12), corresponding to $\mathrm{d}a/\mathrm{d}t = \mathrm{d}\gamma/\mathrm{d}t = 0$, are given by

$$R^2 + (Q_2 - Q_1)R - Q_1 Q_2 = 0, \tag{13}$$

where

$$a_0^2 = \frac{8\omega}{3\xi} R,$$

$$Q_1 = -\rho^2 \left(\frac{1}{32\omega^3} + \frac{1}{8\omega\upsilon(2\omega + \upsilon)} \right) + \rho \frac{4\omega - \upsilon}{8\omega^2} + \frac{\upsilon - 2\omega}{2}$$

and

$$Q_2 = \rho^2 \left(\frac{1}{32\omega^3} + \frac{1}{8\omega\upsilon(2\omega + \upsilon)} \right) + \rho \frac{4\omega - \upsilon}{8\omega^2} - \frac{\upsilon - 2\omega}{2}.$$

Eq. (13) gives a real solution when the following conditions are satisfied

$$\Delta > 0,$$

$$Q_1 > 0, \quad Q_2 < 0. \tag{14}$$

The quantity $\Delta = (Q_2 + Q_1)^2$ is the discriminant of Eq. (13). In Fig. 1, we show the corresponding regions in the parameter space delimited by transition curves for the parameter values $\omega = 1.1$. In the hatched regions, there exist two centers and one saddle denoted, respectively, by C_0, the trivial center, and C_2 and S_2, the cycles of order 2. In the unhatched region only one center exists.

3. Second reduced system and quasi-periodic solutions

To derive the second reduced RS and determine an explicit approximation of QP solutions to the original equation (1), we construct an expansion of the periodic solutions of (12) close to a stationary solution of the unperturbed system of (12), C_2. In [1], a linear approach was proposed to approximate periodic solutions to a first RS system for another type of oscillator. Here we follow the strategy proposed in [2] to obtain relevant non-linear approximate solutions. To appreciate how accurate is the non-linear approach, we refer to Ref. [2] in which comparisons to the linear approach and to numerical integration are given, for a different type of equation.

To apply the multiple scales method to Eq. (12), we introduce the variable change $u = a\cos(\gamma)$, $v = -a \sin(\gamma)$ to transform system (12) to the equivalent Cartesian form

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\mu\frac{\tilde{\alpha}}{2}u + Q_1 v - \frac{3\xi}{8\omega}(u^2 + v^2)v - \mu\frac{\tilde{h}}{2\omega}\cos(\Omega t)v,$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\mu\frac{\tilde{\alpha}}{2}v + Q_2 u + \frac{3\xi}{8\omega}(u^2 + v^2)u + \mu\frac{\tilde{h}}{2\omega}\cos(\Omega t)u. \tag{15}$$

A second-order uniform approximate periodic solution of system (15), in the vicinity of a stationary regime is sought in the form

$$u(t;\mu) = u_0 + \mu u_1(T_0, T_1, T_2) + \mu^2 u_2(T_0, T_1, T_2) + \cdots,$$

$$v(t;\mu) = v_0 + \mu v_1(T_0, T_1, T_2) + \mu^2 v_2(T_0, T_1, T_2) + \cdots, \tag{16}$$

where $T_n = \mu^n t$ and (u_0, v_0) denote the co-ordinates of the fixed point C_2 of the unperturbed system of (12). Substituting Eqs. (16) into system (15) and equating coefficients of same powers of μ, one obtains up to third-order the following systems:

$$D_0^2 u_1 + \Omega_1^2 u_1 = -\frac{\tilde{\alpha}}{2} S^{-1} v_0 + \frac{\tilde{h}}{2\omega} S^{-1} u_0 \cos(\Omega t) + \frac{\tilde{h}}{2\omega}\Omega v_0 \sin(\Omega t), \tag{17}$$

$$v_1 = S\left(D_0 u_1 + \frac{\tilde{\alpha}}{2}u_0 + \frac{\tilde{h}}{2\omega} v_0 \cos(\Omega t) \right), \tag{18}$$

$$D_0^2u_2 + \Omega_1^2 u_2 = -D_0D_1u_1 - S^{-1}D_1v_1 - \left(\frac{\tilde{\alpha}}{2} + \frac{3\xi}{4\omega}(u_0v_1 + v_0u_1)\right)D_0u_1 - \left(\frac{\tilde{h}}{2\omega}\cos(\Omega t) + \frac{3\xi}{4\omega}(u_0u_1 + 3v_0v_1)\right)D_0v_1 + \left(\frac{\tilde{h}}{2\omega}\Omega\sin(\Omega t) - \frac{\tilde{\alpha}}{2}S^{-1}\right)v_1 + \frac{3\xi}{8\omega}S^{-1}(3u_0u_1^2 + 2v_0u_1v_1 + u_0v_1^2) + \frac{\tilde{h}}{2\omega}S^{-1}\cos(\Omega t)u_1, \quad (19)$$

$$v_2 = S\left(D_0u_2 + D_1u_1 + \frac{\tilde{\alpha}}{2}u_1 + \frac{3\xi}{8\omega}(2u_0u_1v_1 + 3v_0v_1^2) + \frac{\tilde{h}}{2\omega}v_1\cos(\Omega t)\right) \quad (20)$$

and

$$D_0^2u_3 + \Omega_1^2u_3 = D_1^2u_1 - 2S^{-1}D_1v_2 - 2S^{-1}D_2v_1 + \left(\tilde{\alpha} + \frac{3\xi}{2\omega}(u_0v_1 + v_0u_1)\right)D_1u_1 + \left(\frac{\tilde{h}}{\omega}\cos(\Omega T_0) + \frac{3\xi}{2\omega}(u_0u_1 + 3v_0v_1)\right)D_1v_1 + \left(\frac{\tilde{\alpha}^2}{4} - \frac{\tilde{h}^2}{4\omega^2}\cos^2(\Omega T_0)\right)u_1 + \frac{\tilde{h}\tilde{\alpha}}{2\omega}\cos(\Omega T_0)v_1 + \tilde{h}R_1\cos(\Omega T_0)u_2 + \left(\frac{\tilde{\alpha}}{2}R_2 + \frac{\tilde{h}\Omega}{2\omega}\sin(\Omega T_0)\right)v_2 + R_3u_1u_2 + R_4v_1u_2 + R_5u_1v_2 + R_6v_1v_2 + \left(\frac{\tilde{\alpha}}{2}R_7 + \tilde{h}R_8\cos(\Omega T_0)\right)u_1v_1 + \left(\frac{\tilde{\alpha}}{2}R_9 + \tilde{h}R_{10}\cos(\Omega T_0)\right)v_1^2 + \left(\frac{\tilde{\alpha}}{2}R_{11} + \tilde{h}R_{12}\cos(\Omega T_0)\right)u_1^2 + R_{13}u_1v_1^2 + R_{14}u_1^3. \quad (21)$$

Here

$$\Omega_1 = \left[\left(-Q_1 + \frac{3\xi}{8\omega}(u_0^2 + 3v_0^2)\right)\left(Q_2 + \frac{3\xi}{8\omega}(v_0^2 + 3u_0^2)\right)\right]^{1/2}$$

is the proper frequency of system (15) and the quantities S and R_i for $i = 1, 2, \dots, 14$ are given in the appendix.

We consider the "satellite" resonance 1 : 2 and we introduce the new detuning parametric σ_1 according to $\Omega = 2\Omega_1 + \sigma_1$ and set $\sigma_1 = \mu^2\tilde{\sigma}_1$ and $\Omega T_0 = 2\Omega_1 T_0 + \tilde{\sigma}_1 T_2$.

Thus, solution of Eq. (17) may be expressed in the following complex form:

$$u_1 = A_1(T_1, T_2)e^{i\Omega_1 T_0} + \tilde{h}(F_2 - iF_3)e^{i\Omega T_0} + \frac{\tilde{\alpha}}{2}F_1 + cc, \quad (22)$$

where A_1 is the slowly varying complex amplitude determined from the higher order expansion. Substituting Eq. (22) into (18), one obtains

$$v_1 = iG_2A_1(T_1, T_2)e^{i\Omega_1 T_0} + \tilde{h}(G_3(F_3 + iF_2) + G_4)e^{i\Omega T_0} + \frac{\tilde{\alpha}}{2}G_1 + cc. \quad (23)$$

Substitution of Eqs. (22) and (23) into Eq. (19) and elimination of the secular terms lead to

$$D_1A_1 = \frac{\tilde{\alpha}A_1}{2}(E_1 - iE_2) + \tilde{h}\bar{A}_1(E_3 + iE_4)e^{i\tilde{\sigma}_1 T_2}. \quad (24)$$

The expressions of E_i, F_i and G_i are given in the appendix. The solution of Eq. (19) is then given by

$$u_2 = \frac{\tilde{\alpha}\tilde{h}}{2}(M_1 + iM_2)e^{i\Omega T_0} + \tilde{h}^2(M_3 + iM_4)e^{2i\Omega T_0} + \tilde{h}A_1(M_5 + iM_6)e^{i(\Omega_1+\Omega)T_0} + A_1^2(M_7 + iM_8)e^{2i\Omega_1 T_0} + \frac{\tilde{\alpha}^2}{4}M_9 + \tilde{h}^2(M_{10} + iM_{11}) + M_{12}A_1\bar{A}_1 + cc. \quad (25)$$

Substituting Eq. (25) into (20) yields

$$v_2 = \left(SD_1A_1 + A_1\frac{\tilde{\alpha}}{2}(N_1 + iN_2)\right)e^{i\Omega_1 T_0} + \frac{\tilde{\alpha}\tilde{h}}{2}(N_3 + iN_4)e^{i\Omega T_0} + A_1^2(N_5 + iN_6)e^{2i\Omega_1 T_0} + \tilde{h}^2(N_7 + iN_8)e^{2i\Omega T_0} + \tilde{h}\bar{A}_1(N_9 + iN_{10})e^{i(\Omega-\Omega_1)T_0} + \tilde{h}A_1(N_{11} + iN_{12})e^{i(\Omega_1+\Omega)T_1} + \frac{\tilde{\alpha}^2}{4}N_{13} + N_{14}A_1\bar{A}_1 + \tilde{h}^2(N_{15} + iN_{16}) + \text{cc}. \tag{26}$$

With the help of Eqs. (22) and (23), the elimination of the secular terms from (21) gives

$$D_2A_1 = \tilde{\alpha}^2A_1(E_5 + iE_6) + \tilde{h}^2A_1(E_7 + iE_8) + A_1^2\bar{A}(E_9 + iE_{10}) + \tilde{\alpha}\tilde{h}\bar{A}_1(E_{11} + iE_{12})e^{i\tilde{\sigma}_1 T_2}. \tag{27}$$

Here M_i and N_i for $i = 1,2,\ldots,12$, and E_i for $i = 5,6,\ldots,12$, are given in the appendix. Eqs. (24) and (27) can be combined to describe the modulation of the complex amplitude to third-order with respect to the original time. Indeed, substituting these equations into expression $\dot{A}_1 = \mu D_1A_1 + \mu^2D_2A_1 + \cdots$, yields

$$\dot{A}_1 = \frac{\alpha A_1}{2}(E_1 - iE_2) + h\bar{A}_1(E_3 + iE_4)e^{i\tilde{\sigma}_1 T_2} + \alpha^2A_1(E_5 + iE_6) + h^2A_1(E_7 + iE_8) + \mu^2A_1^2\bar{A}_1(E_9 + iE_{10}) + \alpha h\bar{A}_1(E_{11} + iE_{12})e^{i\tilde{\sigma}_1 T_2}. \tag{28}$$

Letting $A_1 = \frac{1}{2}a_s e^{i\beta_s}$ where a_s and β_s are real, substituting (27) into Eq. (28), separating real and imaginary parts, we obtain the second reduced modulated autonomous RS

$$\frac{1}{2}\frac{da_s}{dt} = \frac{\alpha a_s}{4}E_1 + \frac{ha_s}{2}(E_3\cos(2\gamma_s) - E_4\sin(2\gamma_s)) + \frac{\alpha^2}{2}E_5a_s + \frac{h^2}{2}E_7a_s + \frac{1}{8}E_{9\mu}a_s^3 + \frac{\alpha h}{2}a_s(E_{11}\cos(2\gamma_s) - E_{12}\sin(2\gamma_s)),$$

$$-\frac{a_s}{2}\frac{d\gamma_s}{dt} = -\frac{a_s}{4}(\Omega - 2\Omega_1) - \frac{\alpha E_2}{4}a_s + \frac{ha_s}{2}(E_3\sin(2\gamma_s) + E_4\cos(2\gamma_s)) + \frac{\alpha^2}{2}E_6a_s + \frac{h^2}{2}E_8a_s + \frac{1}{8}E_{10\mu}a_s^3 + \frac{\alpha h}{2}a_s(E_{11}\sin(2\gamma_s) + E_{12}\cos(2\gamma_s)),$$

$$\gamma_s = \frac{\sigma_1}{2}t - \beta_s, \tag{29}$$

where $E_{9\mu} = \mu^2E_9$ and $E_{10\mu} = \mu^2E_{10}$.

Hence, the problem of investigating periodic solutions of Eq. (15) is transformed to the study of the stationary solutions of this second RS (29) given by setting

$$\frac{da_s}{dt} = 0 \quad \text{and} \quad \frac{d\gamma_s}{dt} = 0.$$

Finally, the approximation up to second order of the periodic solutions of system (12) takes the form

$$u(t) = u_0 + a_s\cos\left(\frac{\Omega t}{2} - \gamma_s\right) + 2h(F_2\cos(\Omega t) + F_3\sin(\Omega t)) + \frac{\alpha}{2}F_1 + \alpha h(M_1\cos(\Omega t) - M_2\sin(\Omega t)) + 2h^2(M_3\cos(2\Omega t) - M_4\sin(2\Omega t)) + ha_s\left(M_5\cos\left(\frac{3}{2}\Omega t - \gamma_s\right) - M_6\sin\left(\frac{3}{2}\Omega t - \gamma_s\right)\right) + \frac{a_s^2}{2}(M_7\cos(\Omega t - 2\gamma_s) - M_8\sin(\Omega t - 2\gamma_s)) + \frac{\alpha^2}{4}M_9 + 2h^2M_{10} + \frac{a_s^2}{4}M_{12}, \tag{30}$$

$$v(t) = v_0 - a_sG_2\sin\left(\frac{\Omega t}{2} - \gamma_s\right) + 2h((G_3F_3 + G_4)\cos(\Omega t) - G_3F_2\sin(\Omega t))$$

$$+ \frac{\alpha}{2} G_1 + \frac{\alpha S}{2} a_s E_1 \cos\left(\frac{1}{2}\Omega t - \gamma_s\right)$$

$$+ \frac{\alpha S}{2} a_s E_2 \sin\left(\frac{1}{2}\Omega t - \gamma_s\right)$$

$$+ hSa_s E_3 \cos\left(\frac{1}{2}\Omega t + \gamma_s\right)$$

$$- hSa_s E_4 \sin\left(\frac{1}{2}\Omega t + \gamma_s\right)$$

$$+ \frac{\alpha a_s}{2}\left(N_1 \cos\left(\frac{1}{2}\Omega t - \gamma_s\right)\right.$$

$$\left. - N_2 \sin\left(\frac{1}{2}\Omega t - \gamma_s\right)\right)$$

$$+ \alpha h(N_3 \cos(\Omega t) - N_4 \sin(\Omega t))$$

$$+ \frac{a_s^2}{2}(N_5 \cos(\Omega t - 2\gamma_s) - N_6 \sin(\Omega t - 2\gamma_s))$$

$$+ 2h^2(N_7 \cos(2\Omega t) - N_8 \sin(2\Omega t))$$

$$+ ha_s\left(N_9 \cos\left(\frac{1}{2}\Omega t + \gamma_s\right)\right.$$

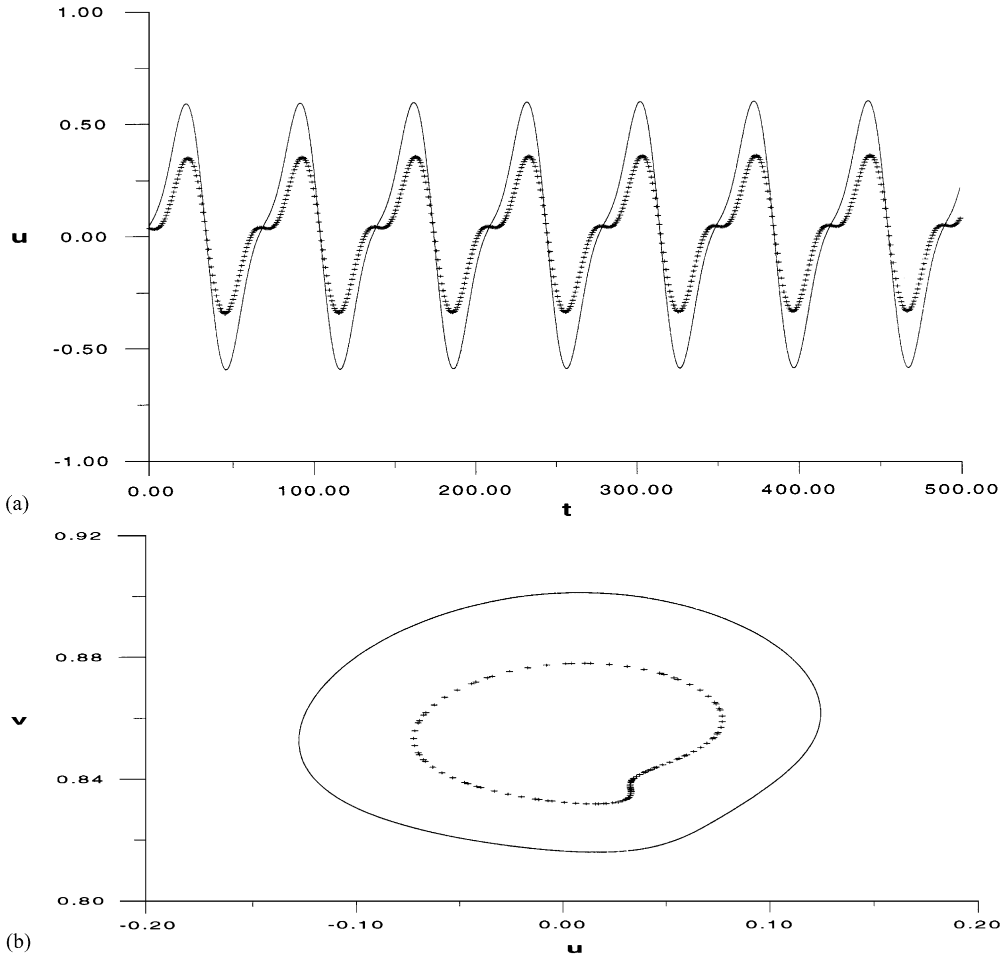

Fig. 2. (a, b) Comparison between numerical and analytical solutions of the reduced system (17) near C_2 in the (u, v) and (t, u) plane, respectively, for $h = 0.01$, $\Omega = 0.181$, $\alpha = 0.001$, $\rho = 0.08$, $\omega = 1.1$, $\xi = 0.5$ and $\upsilon = 2.415$ (++++) numerical simulation, (——) analytical solution of this work.

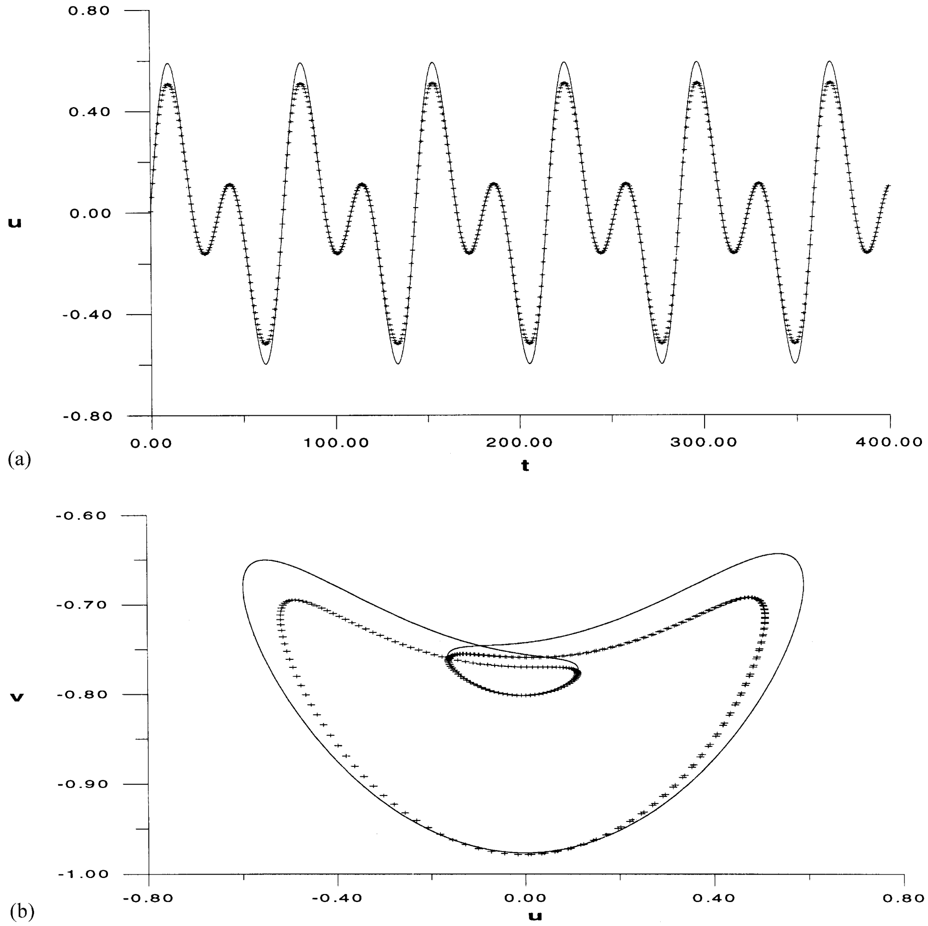

Fig. 3. (a, b) Comparison between numerical and analytical solutions of the reduced system (17) near C_2 in the (u,v) and (t,u) plane, respectively, for $h=0.117$, $\Omega=0.175$, $\alpha=0.001$, $\rho=0.08$, $\omega=1.1$, $\xi=0.5$ and $\upsilon=2.415$. (++++) numerical simulation, (——) analytical solution of this work.

$$
\left.\left. - N_{10}\sin\left(\frac{1}{2}\Omega t+\gamma_s\right)\right)\right) + ha_s\left(N_{11}\cos\left(\frac{3}{2}\Omega t-\gamma_s\right) - N_{12}\sin\left(\frac{3}{2}\Omega t-\gamma_s\right)\right)\Bigg) + \frac{\alpha^2}{4}N_{13}+\frac{a_s^2}{4}N_{14}+2h^2N_{15}. \tag{31}
$$

Combining Eqs. (7), (10), and (30), (31), we obtain an explicit second-order approximation of QP solutions of Eq. (1)

$$
x(t)=u(t)\cos\left(\frac{\upsilon t}{2}\right)-v(t)\sin\left(\frac{\upsilon t}{2}\right)+\frac{\rho}{2\upsilon(2\omega+\upsilon)} \times\left(u(t)\cos\left(\frac{3\upsilon t}{2}\right)-v(t)\sin\left(\frac{3\upsilon t}{2}\right)\right), \tag{32}
$$

where $u(t)$ and $v(t)$ are given by Eqs. (30) and (31).

In Figs. 2 and 3, we report the time history of u and the (u,v) phase portrait for different parameter values of Ω. Comparisons between the analytical approximation of periodic solutions, Eqs. (30) and

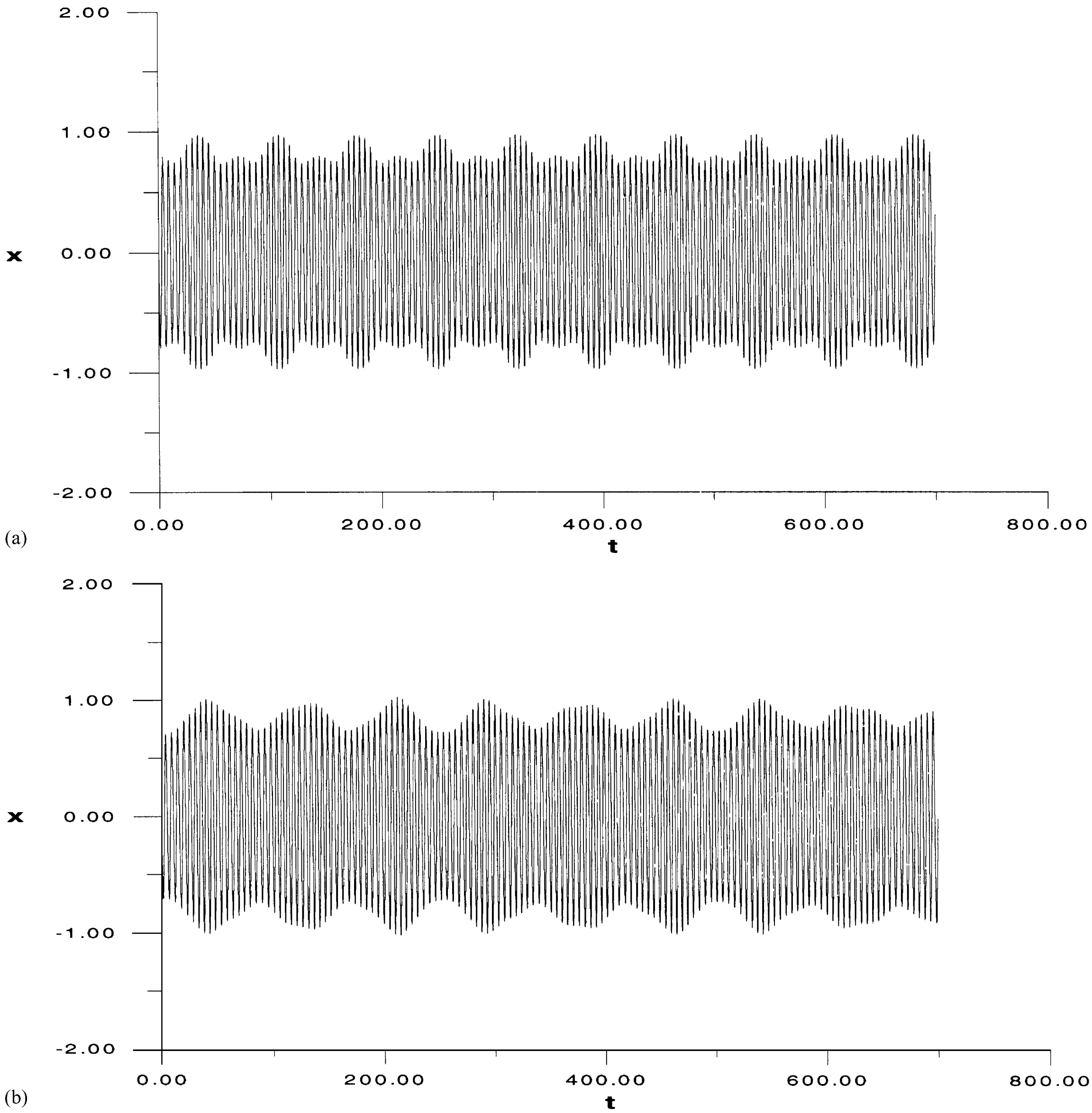

Fig. 4. (a, b) Analytical and numerical solutions of the original system (1) near C_2 in the (t,x) plane for $h=0.117$, $\Omega=0.175$, $\alpha=0.001$, $\rho=0.08$, $\omega=1.1$, $\xi=0.5$ and $\upsilon=2.415$.

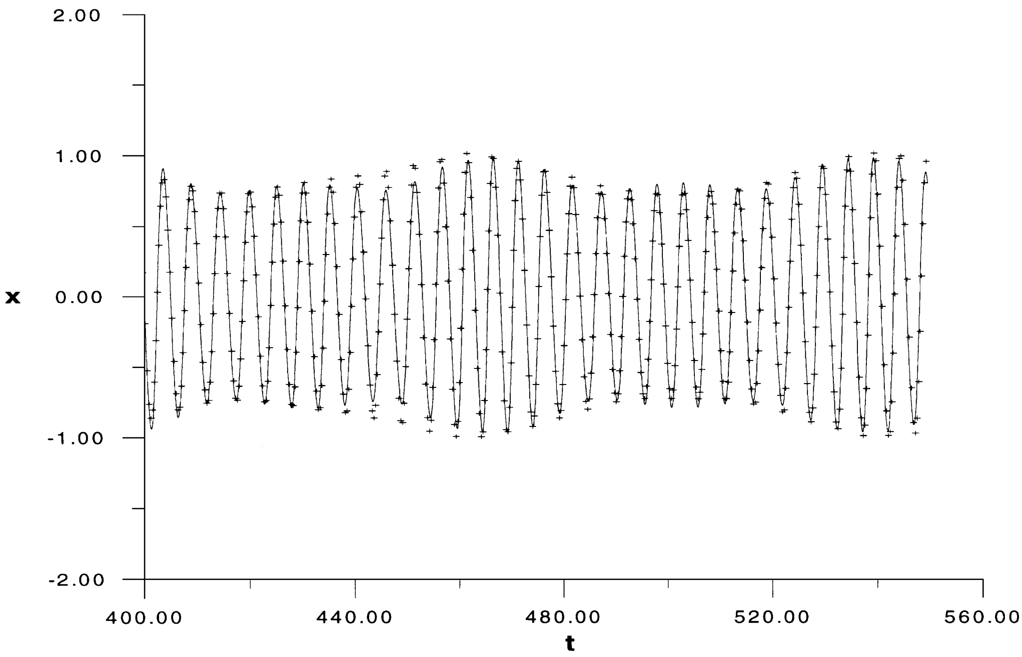

Fig. 5. (a, b) Comparison between numerical and analytical solutions of the original system (1) near C_2 in the (t, x) plane for $h = 0.117$, $\Omega = 0.175$, $\alpha = 0.001$, $\rho = 0.08$, $\omega = 1.1$, $\xi = 0.5$ and $\upsilon = 2.415$. (++++) numerical simulation, (——) analytical solution of this work.

(31), and the direct numerical integration of the first RS system (15) are shown. It can be seen in these figures that the effect of the frequency Ω on the solution is substantial for increasing values of the amplitude h.

In Figs. 4 and 5, we report in time history plane a comparison between the analytical QP solution, Eq. (32), of the original system (1) and the direct numerical integration of the original system (1) using a Runge–Kutta method. Fig. 4a presents numerical integration and Fig. 4b corresponds to the analytical approach. A direct comparison of the two results is given in Fig. 5.

Finally, in Figs. 6 and 7, we report the same comparison of solutions but for a different value of Ω.

4. Conclusion

A double multiple scales method for approximating analytical QP solutions to a weakly damped non-linear QP Mathieu oscillator near the double primary parametric resonance was performed. The first multiple scales method is performed to reduce the original QP equation to an amplitude-phase system having periodic components. The second multiple scales method is used to seek approximate periodic solutions of the latter system corresponding to the QP oscillations of the original one. A very good agreement is obtained between the analytical approximate QP solution of the original equation and its direct numerical integration from both qualitative and quantitative point of view. The concordance of the results is obtained for both the amplitude and the phase of the QP oscillations.

The double reduction procedure presented in the present work has the advantage to realise non-linear approximation of QP solutions. It provides an efficient analytical tool to investigate such solutions for weakly damped non-linear QP oscillators.

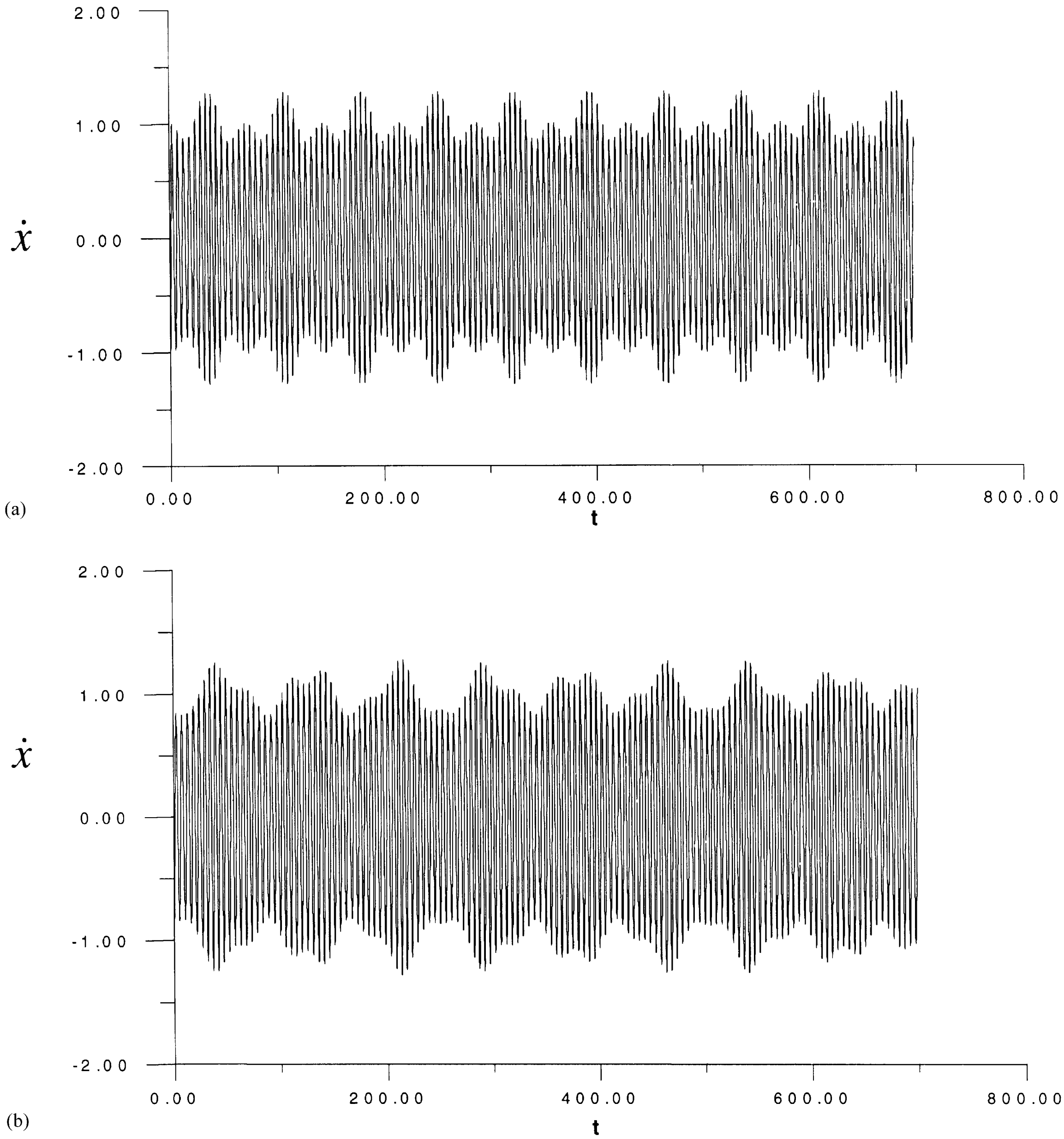

Fig. 6. (a, b) Analytical and numerical solutions of the original system (1) near C_2 in the $(t, \dot{x})$ plane for $h = 0.117$, $\Omega = 0.175$, $\alpha = 0.001$, $\rho = 0.08$, $\omega = 1.1$, $\xi = 0.5$ and $\upsilon = 2.415$.

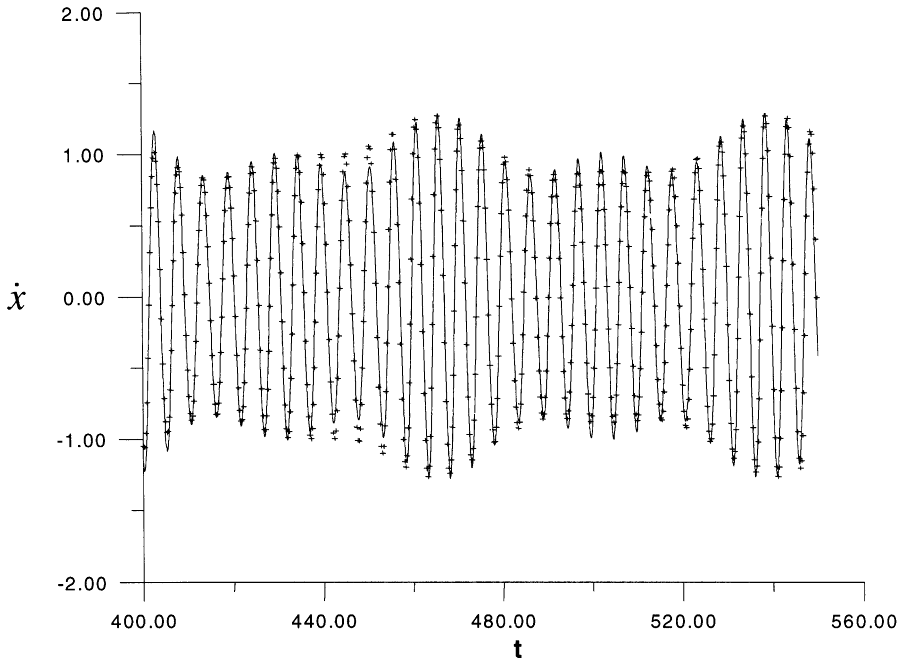

Fig. 7. (a, b) Comparison between numerical and analytical solutions of the original system (1) near C_2 in the $(t, \dot{x})$ plane for $h=0.117$, $\Omega=0.175$, $\alpha=0.001$, $\rho=0.08$, $\omega=1.1$, $\xi=0.5$ and $\upsilon=2.415$. (++++) numerical simulation, (——) analytical solution of this work.

Appendix

$$S=\left(Q_1-\frac{3\xi}{8\omega}(u_0^2+3v_0^2)\right)^{-1},$$

$$S'=\left(Q_2-\frac{3\xi}{8\omega}(3u_0^2+3v_0^2)\right)^{-1},$$

$$F_1=-\frac{v_0}{\Omega_1^2}\left(Q_1-\frac{9\xi}{8\omega}v_0^2\right),$$

$$F_2=-\frac{u_0}{12\omega\Omega_1^2}\left(Q_1-\frac{3\xi}{8\omega}u_0^2\right),\quad F_3=-\frac{v_0}{6\omega\Omega_1},$$

$$G_1=u_0S,\quad G_2=\Omega_1 S,\quad G_3=2\Omega_1 S,\quad G_4=\frac{v_0}{4\omega}S,$$

$$K_1=Q_1-\frac{3\xi}{4\omega}(u_0^2+3v_0^2),$$

$$K_2=Q_1-Q_2-\frac{3\xi}{4\omega}(3u_0^2+v_0^2),$$

$$K_3=-\frac{3\xi}{8\omega}u_0\left(Q_1-\frac{3\xi}{8\omega}u_0^2\right),$$

$$K_4=\frac{9\xi}{8\omega}u_0\left(Q_1-\frac{2}{3}Q_2-\frac{9\xi}{8\omega}u_0^2\right),$$

$$K_5=-\frac{9\xi}{4\omega}v_0\left(Q_2+\frac{3\xi}{8\omega}v_0^2\right),$$

$$E_1=-\frac{K_1G_2}{\Omega_1}+\frac{K_3G_1G_2}{\Omega_1}+\frac{F_1G_2K_5}{2\Omega_1},$$

$$E_2=\frac{F_1K_4}{\Omega_1}+\frac{K_5G_1}{2\Omega_1},$$

$$E_3=-\frac{F_3K_4}{\Omega_1}-\frac{K_3G_2}{\Omega_1}(G_3F_3+G_4)+\frac{F_2G_3K_5}{2\Omega_1}-\frac{F_2G_2K_5}{2\Omega_1},$$

$$E_4=\frac{G_2}{4\omega}-\frac{K_2}{8\omega\Omega_1}-\frac{K_4F_2}{\Omega_1}-\frac{G_2F_2G_3K_3}{\Omega_1}-\frac{K_5}{2\Omega_1}(G_3F_3+G_4)+\frac{F_3G_2K_5}{2\Omega_1},$$

$$L_1 = \frac{v_0}{2\omega} - 2K_1(G_3F_3 + G_4) + \frac{1}{4\omega}F_1K_2 + 2K_3G_1(G_3F_3 + G_4) + 2F_1F_2K_4 + F_2G_1K_5 + F_1K_5(G_3F_3 + G_4),$$

$$L_2 = -2K_1F_2G_3 - \frac{\Omega_1}{2\omega}G_1 + 2F_2G_1G_3K_3 - 2K_4F_1F_3 - F_3G_1K_5 + F_1F_2G_3K_5,$$

$$L_3 = -\frac{u_0}{16\omega^2} + \frac{\Omega_1}{2\omega}G_3F_2 + \frac{1}{4\omega}K_2F_2 + K_3(G_3^2(F_3^2 - F_2^2) + G_4^2 + 2G_3F_3G_4) + K_4(F_2^2 - F_3^2) + K_5(F_2F_3G_3 + F_2G_4 + F_2F_3G_3),$$

$$L_4 = -\frac{1}{2\omega}\Omega_1(G_3F_3 + G_4) - \frac{1}{4\omega}F_3K_2 + K_3(2F_2F_3G_3^2 + 2F_2G_3G_4 - 2K_4F_2F_3 + K_5(G_3F_2^2 - F_3(G_3F_3 + G_4))),$$

$$L_5 = \frac{1}{2\omega}\Omega_1G_2 + \frac{1}{4\omega}K_2 - 2F_2G_2G_3K_3 + 2K_4F_2 + K_5(G_3F_3 + G_4) + G_2F_3K_5,$$

$$L_6 = 2K_3G_2(G_3F_3 + G_4) - 2K_4F_3 + F_2G_3K_5 + F_2G_2K_5,$$

$$L_7 = K_4 - K_3G_2^2, \quad L_8 = K_5G_2,$$

$$L_9 = u_0 - 2G_1K_1 + K_3G_1^2 + F_1^2K_4 + G_1F_1K_5,$$

$$L_{10} = -\frac{u_0}{8\omega^2} - \frac{1}{2\omega}\Omega_1G_3F_2 + \frac{1}{4\omega}F_2K_2 + 2K_3(G_3^2(F_3^2 + F_2^2) + G_4^2 + 2G_3F_3G_4) + 2K_4(F_3^2 + F_2^2) + K_5(F_2F_3G_3 + F_2G_4 - F_2F_3G_3),$$

$$L_{11} = \frac{1}{2\omega}\Omega_1(G_3F_3 + G_4) - \frac{1}{4\omega}F_3K_2 + K_5(G_3F_2^2 + F_3(G_3F_3 + G_4)),$$

$$L_{12} = K_4 + K_3G_2^2,$$

$$M_1 = L_1(-3\Omega_1^2)^{-1}, \quad M_2 = L_2(-3\Omega_1^2)^{-1},$$

$$M_3 = L_3(-15\Omega_1^2)^{-1}, \quad M_4 = L_4(-15\Omega_1^2)^{-1},$$

$$M_5 = -L_5(8\Omega_1^2)^{-1}, \quad M_6 = -L_6(8\Omega_1^2)^{-1},$$

$$M_7 = -L_7(3\Omega_1^2)^{-1}, \quad M_8 = -L_8(3\Omega_1^2)^{-1},$$

$$M_9 = L_9(\Omega_1^2)^{-1}, \quad M_{10} = L_{10}(\Omega_1^2)^{-1},$$

$$M_{11} = L_{11}(\Omega_1^2)^{-1}, \quad M_{12} = L_{12}(2\Omega_1^2)^{-1},$$

$$N_1 = S\left(1 + \frac{3\xi}{4\omega}u_0G_1 + \frac{3\xi}{4\omega}v_0F_1\right),$$

$$N_2 = S\left(\frac{3\xi}{4\omega}u_0F_1G_2 + \frac{9\xi}{4\omega}v_0G_1G_2\right),$$

$$N_3 = S\left(-2\Omega_1M_2 + F_2 + \frac{1}{4\omega}G_1 + \frac{3\xi}{4\omega}u_0F_2G_1 + \frac{3\xi}{4\omega}u_0F_1(G_3F_3 + G_4) + \frac{9\xi}{4\omega}v_0G_1(G_3F_3 + G_4) + \frac{3\xi}{4\omega}v_0F_1F_2\right),$$

$$N_4 = S\left(2\Omega_1M_1 - F_3 - \frac{3\xi}{4\omega}u_0F_3G_1 + \frac{3\xi}{4\omega}u_0F_1G_3F_2 + \frac{9\xi}{4\omega}v_0G_1G_3F_2 - \frac{3\xi}{4\omega}v_0F_1F_3\right),$$

$$N_5 = S\left(-2\Omega_1M_8 - \frac{9\xi}{8\omega}v_0G_2^2 + \frac{3\xi}{8\omega}v_0\right),$$

$$N_6 = S\left(2\Omega_1M_7 + \frac{3\xi}{4\omega}u_0G_2\right),$$

$$N_7 = S\left(-4\Omega_1M_4 + \frac{1}{4\omega}(G_3F_3 + G_4) + \frac{3\xi}{4\omega}u_0(F_2(G_3F_3 + G_4) + F_3G_3F_2)\right.$$

$$+\frac{9\xi}{8\omega}v_0(G_3^2(F_3^2-F_2^2)+G_4^2+2F_3G_3G_4)$$

$$\left.+\frac{3\xi}{8\omega}v_0(F_2^2-F_3^2)\right),$$

$$N_8=S\left(4\Omega_1M_3+\frac{1}{4\omega}G_3F_2\right.$$

$$+\frac{3\xi}{4\omega}u_0(-F_3(G_3F_3+G_4)$$

$$+F_2^2G_3)+\frac{9\xi}{8\omega}v_0(2G_3^2F_2F_3$$

$$\left.+2F_2G_3G_4)-\frac{3\xi}{4\omega}v_0F_2F_3\right),$$

$$N_9=S\left(\frac{3\xi}{4\omega}u_0(G_3F_3+G_4)-\frac{3\xi}{4\omega}u_0F_3G_2\right.$$

$$\left.+\frac{9\xi}{4\omega}v_0F_2G_3G_2+\frac{3\xi}{4\omega}v_0F_2\right),$$

$$N_{10}=S\left(-\frac{1}{4\omega}G_2+\frac{3\xi}{4\omega}u_0F_2G_3-\frac{3\xi}{4\omega}u_0F_2G_2\right.$$

$$\left.-\frac{9\xi}{4\omega}v_0G_2(G_3F_3+G_4)-\frac{3\xi}{4\omega}v_0F_3\right),$$

$$N_{11}=S\left(-3\Omega_1M_6+\frac{3\xi}{4\omega}u_0(G_3F_3+G_4)\right.$$

$$\left.+\frac{3\xi}{4\omega}u_0F_3G_2-\frac{9\xi}{4\omega}v_0F_2G_3G_2+\frac{3\xi}{4\omega}v_0F_2\right),$$

$$N_{12}=S\left(3\Omega_1M_5+\frac{1}{4\omega}G_2+\frac{3\xi}{4\omega}u_0G_3F_2\right.$$

$$+\frac{3\xi}{4\omega}u_0F_2G_2+\frac{9\xi}{4\omega}v_0G_2(G_3F_3+G_4)$$

$$\left.-\frac{3\xi}{4\omega}v_0F_3\right),$$

$$N_{13}=S\left(F_1+\frac{3\xi}{4\omega}u_0G_1F_1+\frac{9\xi}{8\omega}v_0G_1^2+\frac{3\xi}{8\omega}v_0F_1^2\right),$$

$$N_{14}=\frac{3\xi}{16\omega}v_0S(3G_2^2+1),$$

$$N_{15}=S\left(\frac{1}{4\omega}(G_3F_3+G_4)+\frac{3\xi}{4\omega}u_0(F_2(G_3F_3+G_4)\right.$$

$$-F_2F_3G_3)+\frac{9\xi}{4\omega}v_0((G_3F_3+G_4)^2+(F_2G_3)^2)$$

$$\left.+\frac{3\xi}{4\omega}v_0(F_2^2+F_3^2)\right),$$

$$N_{16}=S\left(\frac{1}{4\omega}G_3F_2+\frac{3\xi}{4\omega}u_0(F_3(G_3F_3\right.$$

$$\left.+G_4)+F_2^2G_3)\right),$$

$$R_1=\frac{1}{2\omega}(S^{-1}-S'^{-1}+\frac{3\xi}{4\omega}(v_0^2-u_0^2)),$$

$$R_2=2S^{-1}+\frac{3\xi}{4\omega}(u_0^2-3v_0^2),$$

$$R_3=\frac{3\xi}{2\omega}u_0\left(\frac{3}{2}S^{-1}-S'^{-1}\right),$$

$$R_4=-\frac{9\xi}{4\omega}S'^{-1}v_0,\quad R_5=-\frac{9\xi}{4\omega}S'^{-1}v_0,$$

$$R_6=-\frac{3\xi}{4\omega}S^{-1}u_0,\quad R_7=\frac{3\xi}{\omega}u_0,\quad R_8=-\frac{3\xi}{4\omega^2}v_0,$$

$$R_9=\frac{9\xi}{2\omega}v_0,\quad R_{10}=-\frac{3\xi}{8\omega^2}u_0,$$

$$R_{11}=\frac{3\xi}{2\omega}v_0,\quad R_{12}=-\frac{9\xi}{8\omega^2}u_0,$$

$$R_{13}=\frac{3\xi}{8\omega}\left(S^{-1}-S'^{-1}+\frac{3\xi}{4\omega}(u_0^2-3v_0^2)\right),$$

$$R_{14}=\frac{3\xi}{8\omega}\left(S^{-1}-S'^{-1}+\frac{3\xi}{4\omega}(v_0^2-3u_0^2)\right),$$

$$I_1=-2S^{-1}N_1+2+\frac{3\xi}{2\omega}u_0G_1+\frac{3\xi}{2\omega}v_0F_1+SR_2$$

$$+SR_5F_1+SR_6G_1,$$

$$I_2=-2S^{-1}N_2+\frac{3\xi}{2\omega}u_0F_1G_2+\frac{9\xi}{2\omega}v_0G_1G_2,$$

$$I_3=-2S^{-1}N_9+\frac{3\xi}{2\omega}u_0(G_3F_3+G_4)+\frac{3\xi}{2\omega}v_0F_2$$

$$-\frac{3\xi}{2\omega}u_0G_2F_3+\frac{9\xi}{2\omega}v_0G_2G_3F_2+SR_5F_2+SR_6(G_3F_3+G_4),$$

$$I_4=-2S^{-1}N_{10}+\frac{3\xi}{2\omega}u_0G_3F_2-\frac{3\xi}{2\omega}v_0F_3-\frac{3\xi}{2\omega}u_0G_2F_2-\frac{9\xi}{2\omega}v_0G_2(G_3F_3+G_4)-\frac{G_2}{2\omega}-\frac{\Omega}{4\omega}S-R_5SF_3+R_6SF_2G_3,$$

$$P_1=\tfrac{1}{4}(1+R_2N_1+R_3M_9+R_5N_{13}+R_5F_1N_1+R_6G_1N_1+R_7G_1+2R_{11}F_1+R_{13}G_1^2+3R_{14}F_1^2),$$

$$P_2=\tfrac{1}{4}(R_2N_2+R_4G_2M_9+R_5F_1N_2+R_6G_2N_{13}+R_6G_1N_2+R_7G_2F_1+2R_9G_1G_2+2R_{13}G_1G_2F_1),$$

$$P_3=-\frac{1}{8\omega^2}-\frac{\Omega}{4\omega}N_{10}-\frac{\Omega}{4\omega}N_{12}+2R_3M_{10}+\frac{R_1}{2}M_5+R_3F_2M_5-R_3F_3M_6+R_4M_5(G_3F_3+G_4)+R_4M_6G_3F_2+2R_5N_{15}+R_5F_2N_9-R_5F_3N_{10}+R_5F_2N_{11}-R_5F_3N_{12}+R_6N_9(G_3F_3+G_4)+R_6N_{10}F_2G_3+R_6N_{11}(G_3F_3+G_4)+R_6N_{12}F_2G_3+R_8(G_3F_3+G_4)+2R_{12}F_2+2R_{13}(G_4^2+2G_3F_3G_4+G_3^2F_3^2+G_3^2F_2^2)+4R_{13}G_2F_3(G_3F_3+G_4)+6R_{14}(F_2^2+F_3^2),$$

$$P_4=-\frac{\Omega}{4\omega^2}N_9+\frac{\Omega}{4\omega}N_{11}+\frac{R_1}{2}M_6+R_3F_3M_5+R_3F_2M_6+R_4G_2M_{10}+R_4M_6(G_3F_3+G_4)-R_4M_5G_3F_2+R_4G_2M_{10}-R_5F_2N_{10}-R_5F_3N_9+R_5F_2N_{12}+R_5F_3N_{11}+2R_6G_2N_{15}+R_6N_9F_2G_3-R_6N_{10}(G_3F_3+G_4)+R_6N_{12}(G_3F_3+G_4)-R_6N_{11}F_2G_3+R_8G_2F_2+2R_{10}G_2(G_3F_3+G_4)+4R_{13}G_2F_2(G_3F_3+G_4),$$

$$P_5=2R_3M_{12}+R_3M_7+R_4G_2M_8+2R_5N_{14}+R_5N_5+G_2R_6N_6+2R_{13}G_2^2+3R_{14}-R_{13}G_2^2,$$

$$P_6=R_3M_8+2R_4M_{12}G_2-R_4G_2M_7+R_5N_6+2R_6G_2N_{14}-G_2R_6N_5,$$

$$P_7=\frac{R_2}{2}N_9-\frac{\Omega}{8\omega}N_2+\frac{R_3}{2}M_1+\frac{R_4}{2}G_2M_2+\frac{R_5}{2}F_1N_9+\frac{R_5}{2}N_3+\frac{R_5}{2}F_2N_1-\frac{R_5}{2}F_3N_2+\frac{R_6}{2}G_1N_9+\frac{R_6}{2}G_2N_4+\frac{R_6}{2}N_1(G_3F_3+G_4)+\frac{R_6}{2}N_2F_2G_3+\frac{R_7}{2}G_2F_3+\frac{R_7}{2}(G_3F_3+G_4)+\frac{R_8}{4}G_1+R_9G_2G_3F_2+R_{11}F_2+\frac{R_{12}}{2}F_1+R_{13}G_2F_1G_3F_2+R_{13}G_1(G_3F_3+G_4)-R_{13}G_1G_2F_3+3R_{14}F_1F_2,$$

$$P_8=-\frac{G_2}{4\omega}+\frac{R_2}{2}N_{10}-\frac{\Omega}{8\omega}N_1+\frac{R_3}{2}M_2+\frac{R_4}{2}G_2M_1+\frac{R_5}{2}F_1N_{10}+\frac{R_5}{2}N_4-\frac{R_5}{2}F_2N_2-\frac{R_5}{2}F_3N_1+\frac{R_6}{2}G_1N_{10}-\frac{R_6}{2}G_2N_3-\frac{R_6}{2}N_2(G_3F_3+G_4)+\frac{R_6}{2}N_1F_2G_3-\frac{R_7}{2}G_2F_2+\frac{R_7}{2}G_3F_2-\frac{R_8}{4}G_2F_1-R_9G_2(G_3F_3+G_4)$$

$$-\frac{R_{10}}{2}G_1G_2 - R_{11}F_3 - R_{13}G_2F_1(G_3F_3 + G_4)$$

$$+R_{13}G_1G_3F_2 - R_{13}G_1G_2F_2 - 3R_{14}F_1F_3,$$

$$Z_1 = \frac{\alpha}{2}E_1 + \alpha^2E_5 + h^2E_7, \quad Z_2 = hE_3 + \alpha hE_{11},$$

$$Z_3 = hE_4 + \alpha hE_{12},$$

$$Z_4 = -\frac{\alpha}{2}E_2 + \alpha^2E_6 + h^2E_8 - \frac{\Omega - 2\Omega_1}{2},$$

$$E_5 = \frac{S}{2G_2}\left(\frac{E_1E_2}{2} + \frac{1}{4}(E_1I_2 - I_1E_2) + P_2\right),$$

$$E_6 = \frac{S}{2G_2}\left(\frac{E_1^2 - E_2^2}{4} - \frac{E_1I_1 + E_2I_2}{4} - P_1\right),$$

$$E_7 = \frac{S}{2G_2}(P_4 + E_3I_4 - E_4I_3),$$

$$E_8 = \frac{S}{2G_2}(E_3^2 + E_4^2 - P_3 - E_3I_4 - E_4I_4),$$

$$E_9 = \frac{S}{2G_2}P_6, \quad E_{10} = \frac{-S}{2G_2}P_5,$$

$$E_{11} = \frac{S}{2G_2}\left(-E_1E_4 + \frac{E_4I_1 + I_2E_3}{2}\right.$$

$$\left. + \frac{E_1I_4 + E_2I_3}{2} + P_8\right),$$

$$E_{12} = \frac{S}{2G_2}\left(E_1E_3 - \frac{E_3I_1 - E_4I_2}{2}\right.$$

$$\left. - \frac{E_1I_3 - E_2I_4}{2} - P_7\right),$$

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