Effect of quasiperiodic gravitational modulation on the stability of a heated fluid layer

Taoufik Boulal, Saïd Aniss,* and Mohamed Belhaq
University Hassan II Aïn-Chock, Faculty of Sciences, Laboratory of Mechanics, Casablanca, Morocco

Richard Rand
Department of Theoretical and Applied Mechanics, Cornell University, Ithaca, New York 14853-1503, USA
(Received 31 August 2007; published 30 November 2007)

Thermal instability in a horizontal Newtonian liquid layer with rigid boundaries is investigated in the presence of vertical quasiperiodic forcing having two incommensurate frequencies ω_1 and ω_2. By means of a Galerkin projection truncated to the first order, the governing linear system corresponding to the onset of convection is reduced to a damped quasiperiodic Mathieu equation. The threshold of convection corresponding to quasiperiodic solutions is determined in the cases of heating from below and heating from above. We show that a modulation with two incommensurate frequencies has a stabilizing or a destabilizing effect depending on the frequencies ratio $\omega=\omega_2/\omega_1$. The effect of the Prandtl number in a stabilizing zone is also examined for different frequency ratios.

DOI: 10.1103/PhysRevE.76.056320 PACS number(s): 47.20.−k, 44.25.+f

I. INTRODUCTION

Several works have been devoted to analyzing the effect of periodic modulation on the stability of the motionless state of a heated liquid layer. This periodic modulation is associated with either gravity or with temperatures imposed on the horizontal planes of a liquid layer. The gravitational modulation, which can be realized by a vertically oscillating horizontal liquid layer, acts on the entire volume of the liquid and may have a stabilizing or destabilizing effect depending on the amplitude and frequency of the forcing [1–13]. This effect was analyzed for a liquid layer heated from below or from above (respectively, stable or unstable equilibrium configurations). Here the onset of convection presents a competition between harmonic and subharmonic modes; see, for instance, Refs. [1,5,6,9–13]. A similar modulation to that of gravity can be realized by heating a horizontal ferromagnetic liquid layer from above and forcing it with a time-periodic external magnetic field [11,12]. The other type of modulation [14–25], in which the temperature is forced to oscillate at the boundaries, is mainly concentrated in a boundary layer whose thickness decreases with increasing frequency of modulation. Note that this modulation is similar to the gravitational one for low modulation frequencies. In the out of-phase oscillations case, the modulation presents a stabilizing effect. However, in the in-phase oscillations case, the modulation presents a stabilizing effect for low frequencies and a destabilizing effect at high frequencies [2].

In contrast to the standard periodic modulation, the present work focuses attention on the influence of a quasiperiodic gravitational modulation on the convective instability threshold. Here we consider a Newtonian fluid layer confined between two horizontal rigid plates of infinite extent and submitted to a vertical quasiperiodic displacement having two incommensurate frequencies. This motion can be realized, for instance, by using a coupled system of two blocks and two springs attached to a horizontal wall and oscillating vertically. One of the two blocks corresponds to the physical system under consideration. Neglecting friction, the solution of the linear system involves a quasi-periodic motion with two incommensurate frequencies. A Galerkin method truncated to first order is implemented to reduce the linear problem of convection to a linear damped quasiperiodic Mathieu oscillator. Note that in this case, Floquet theory cannot be applied to determine a stability criterion. However, the recent works by Rand *et al.* [26,27] provide a stability chart for the quasiperiodic Mathieu oscillator. This result is used to investigate the quasiperiodic parametric instability in our specific physical problem and to determine a convective instability criterion as a function of the various parameters of the problem. This technique is based principally on deriving approximate analytical expressions for marginal stability curves using the method of harmonic balance and Hill's determinants. In contrast to the periodic modulation case where the onset of convection corresponds to harmonic or subharmonic solutions, here the threshold of convection corresponds precisely to quasiperiodic solutions.

II. FORMULATION

Consider a Newtonian fluid bounded between two horizontal plates having, respectively, constant temperatures T_o at $z=-\frac{d}{2}$ and T_1 at $z=\frac{d}{2}$ ($T_o>T_1$ or $T_o<T_1$). Assume that the fluid layer is submitted to vertical quasiperiodic motion according to the law of displacement

$$z=b_1\cos(\omega_1 t)+b_2\cos(\omega_2 t),$$

where ω_1 and ω_2 are two incommensurate frequencies. The parameters b_1 and b_2 are the amplitudes of motion. Therefore, the fluid layer is submitted to two volumic forces: the gravitational force $\rho\mathbf{g}$ and the quasiperiodic one $-\rho[b_1\omega_1^2\cos(\omega_1 t)+b_2\omega_2^2\cos(\omega_2 t)]\mathbf{k}$. We denote by $\mathbf{k}$, the unit vector upward. The equilibrium of the fluid layer corresponds to a rest state with a conductive regime. Under these assumptions, the linear system of the governing equa-

*s.aniss@fsac.ac.ma

 ©2007 The American Physical Society

tions, corresponding to a perturbation of equilibrium state, is given by the following Navier-Stokes equations in the Boussinesq approximation

$$\nabla \cdot \mathbf{V} = 0, \tag{1}$$

$$\rho \frac{\partial \mathbf{V}}{\partial t} = -\nabla P + \mu \Delta \mathbf{V} + \rho \beta [g + b_1 \omega_1^2 \cos(\omega_1 \ t) + b_2 \omega_2^2 \cos(\omega_2 \ t)] T \ \mathbf{k}, \tag{2}$$

$$\frac{\partial T}{\partial t} - \frac{T_o - T_1}{d} w = \kappa \Delta T, \tag{3}$$

where ρ, β, μ, and κ designate, respectively, the density, the coefficient of thermal dilatation, the dynamic viscosity, and the thermal diffusivity of the fluid. In this study, we assume that $b_1\omega_1^2 = b_2\omega_2^2$. Scaling time by $\frac{d^2}{\kappa}$, the coordinates by d, the velocity field by $\frac{\kappa}{d}$, the pressure by $\frac{\rho \ \nu\kappa}{d^2}$, and the temperature by $(T_o - T_1)$, we obtain the following dimensionless system corresponding to a linear perturbation of the basic state

$$\left[\mathrm{Pr}^{-1}\frac{\partial}{\partial t} - \Delta\right]\Delta w - \mathrm{Ra}\{1 + \alpha[\cos(\Omega_1 t) + \cos(\Omega_2 t)]\}\Delta_2 T = 0, \tag{4}$$

$$\frac{\partial T}{\partial t} - w = \Delta T. \tag{5}$$

Equation (4) represents the vertical component of the vorticity where $\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, $\mathrm{Ra} = \frac{\rho g \beta (T_0 - T_1) d^3}{\mu\kappa}$ is the gravitational Rayleigh number, $\mathrm{Pr} = \frac{\nu}{\kappa}$ is the Prandtl number, $\Omega_1 = \frac{d^2\omega_1}{\kappa}$ and $\Omega_2 = \frac{d^2\omega_2}{\kappa}$ are two dimensionless incommensurate frequencies. We denote the amplitude ratio of the acceleration of the oscillatory motion to the acceleration of gravity by $\alpha = \frac{b_1\omega_1^2}{g} = \frac{b_2\omega_2^2}{g}$. This coefficient can be written as $\alpha = \mathrm{Fr}_1\Omega_1^2 = \mathrm{Fr}_2\Omega_2^2$, where Fr_1 and Fr_2 are two Froude numbers defined by

$$\mathrm{Fr}_1 = \frac{(\kappa/d)^2}{gd}\frac{b_1}{d}, \quad \mathrm{Fr}_2 = \frac{(\kappa/d)^2}{gd}\frac{b_2}{d}.$$

The boundary conditions on temperature and velocity, for the rigid-rigid case, are given by

$$T = w = \frac{\partial w}{\partial z} = 0 \quad \text{at} \quad z = \mp \frac{1}{2}. \tag{6}$$

III. STABILITY ANALYSIS

Using the normal mode analysis, the third component of velocity w and the temperature T are written in the form

$$w = w_1(z,t)\exp(iq_x x + iq_y y),$$

$$T = T_1(z,t)\exp(iq_x x + iq_y y). \tag{7}$$

Here, q_x and q_y represent the wave numbers in the x and y directions. Introducing Eq. (7) into the system (4) and (5), one obtains

$$\left[\mathrm{Pr}^{-1}\frac{\partial}{\partial t} - (D^2 - q^2)\right][D^2 - q^2]w_1 + q^2\mathrm{Ra}\{1 + \mathrm{Fr}_1\Omega_1^2[\cos(\Omega_1 t) + \cos(\Omega_2 t)]\}T_1 = 0, \tag{8}$$

$$\left[\frac{\partial}{\partial t} - (D^2 - q^2)\right]T_1 - w_1 = 0, \tag{9}$$

where $q^2 = q_x^2 + q_y^2$ and $D = \partial/\partial z$. To solve the system (8) and (9), together with the boundary conditions (6), we seek a solution by means of a first order Galerkin method

$$w_1(z,t) = g(t)Z_1(z), \quad T_1(z,t) = f(t)Z_2(z)$$

with

$$Z_1(z) = \left(z^2 - \frac{1}{4}\right)^2, \quad Z_2(z) = \left(z^2 - \frac{1}{4}\right)\left(\frac{5}{4} - z^2\right).$$

Note that the trial functions $Z_1(z)$ and $Z_2(z)$ are used first by Gershuni [2] in the classical Rayleigh-Bénard problem and represent a good approximation to determine the convection threshold. Indeed, these trial functions lead to the critical Rayleigh number 1717.98. The exact value given by Chandrasekhar is 1707.8 [28]. Under these assumptions and applying the Galerkin method, the system (8) and (9) is reduced to the damped quasi-periodic amplitude equation of temperature

$$\frac{d^2 f}{dt^2} + 2p\frac{df}{dt} + c\{R_0 - \mathrm{Ra}[1 + Fr_1\Omega_1^2(\cos(\Omega_1 t) + \cos(\Omega_2 t))]\}f = 0, \tag{10}$$

where $R_0 = \frac{4(q^4 + 24q^2 + 504)(306 + 31q^2)}{121 \ q^2}$ is the gravitational Rayleigh number of the marginal stability curve for the classical Rayleigh-Bénard problem. The coefficients p and c are given by

$$2p = \left(\frac{306}{31} + q^2\right) + \mathrm{Pr}\frac{q^4 + 24q^2 + 504}{12 + q^2}, \quad c = \frac{121 \ \mathrm{Pr} \ q^2}{124(12 + q^2)}.$$

Using the change of variable $\tau = \Omega_1 t$, we obtain a similar equation to the one studied by Rand *et al.* [26,27]

$$\frac{d^2 f}{d\tau^2} + 2\mu\frac{df}{d\tau} + \{\delta + \epsilon[\cos(\tau) + \cos(\omega\tau)]\}f = 0, \tag{11}$$

where $\mu = \frac{p}{\Omega_1}$, $\delta = -\frac{c(\mathrm{Ra} - R_0)}{\Omega_1^2}$, $\epsilon = -c\mathrm{Fr}_1\mathrm{Ra}$, and $\omega = \frac{\Omega_2}{\Omega_1}$.

As noted above, Floquet theory cannot be used to determine solutions of Eq. (11). Following Rand *et al.* [26,27], we use the harmonic balance method to determine the marginal stability curves by means of expansion

$$f(t) = \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\left[A_{nm}\cos\left(\frac{n + m\omega}{2}\tau\right) + B_{nm}\sin\left(\frac{n + m\omega}{2}\tau\right)\right] \tag{12}$$

in which we may set, without loss of generality, $A_{-n,-m} = A_{n,m}$ and $B_{-n,-m} = -B_{n,m}$. Approximate results are obtained by a truncation of the infinite sums in Eq. (12) and then replaced by sums from 0 to N for n and from $-N$ to N for m, respectively. In the case, $N=1(n=0,1;\ m=-1,0,1)$, Eqs. (11) and (12) allow us to obtain two homogenous algebraic systems in A_{nm} and B_{nm}. The first system in A_{nm} is of the form

$$\left(\delta-\frac{\omega^2}{4}\right)A_{0,1}+\mu\omega B_{0,1}+\frac{\epsilon}{2}A_{0,1}=0,\quad \delta A_{0,0}=0,$$

$$\left(\delta-\frac{(1-\omega)^2}{4}\right)A_{1,-1}+\mu(1-\omega)B_{1,-1}+\epsilon A_{1,1}=0,$$

$$\left(\delta-\frac{1}{4}+\frac{\epsilon}{2}\right)A_{1,0}+\mu B_{1,0}=0,$$

$$\left(\delta-\frac{(1+\omega)^2}{4}\right)A_{1,1}+\mu(1+\omega)B_{1,1}+\epsilon A_{1,-1}=0.$$

The second system in B_{nm} is given by

$$\left(\delta-\frac{\omega^2}{4}-\frac{\epsilon}{2}\right)B_{0,1}-\mu\omega A_{0,1}=0,$$

$$\left(\delta-\frac{(1-\omega)^2}{4}\right)B_{1,-1}-\mu(1-\omega)A_{1,-1}=0,$$

$$\left(\delta-\frac{1}{4}-\frac{\epsilon}{2}\right)B_{1,0}-\mu A_{1,0}=0,\quad \left(\delta-\frac{(1+\omega)^2}{4}\right)B_{1,1}-\mu(1+\omega)A_{1,1}=0.$$

Algebraically eliminating the $B_{n,m}$ from these two systems, we obtain the following system in $A_{n,m}$:

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 \\ 0 & 0 & a_{33} & 0 & a_{35} \\ 0 & 0 & 0 & a_{44} & 0 \\ 0 & 0 & a_{53} & 0 & a_{55} \end{pmatrix}\begin{pmatrix} A_{0,0} \\ A_{0,1} \\ A_{1,-1} \\ A_{1,0} \\ A_{1,1} \end{pmatrix}=0, \qquad (13)$$

where

$$a_{11}=\delta,\quad a_{22}=-\frac{\frac{\epsilon^2}{2}-\left(\delta-\frac{\omega^2}{4}\right)^2-\mu^2\omega^2}{\frac{\omega^2}{4}-\delta+\frac{\epsilon}{2}},$$

$$a_{33}=\frac{\left(\delta-\frac{(1-\omega)^2}{4}\right)^2+\mu^2(1-\omega)^2}{\delta-\frac{(1-\omega)^2}{4}},\quad a_{35}=\epsilon,$$

$$a_{53}=\epsilon,\quad a_{44}=\frac{\left(\delta-\frac{1}{4}\right)^2-\frac{\epsilon^2}{4}+\mu^2}{\delta-\frac{1}{4}-\frac{\epsilon}{2}},$$

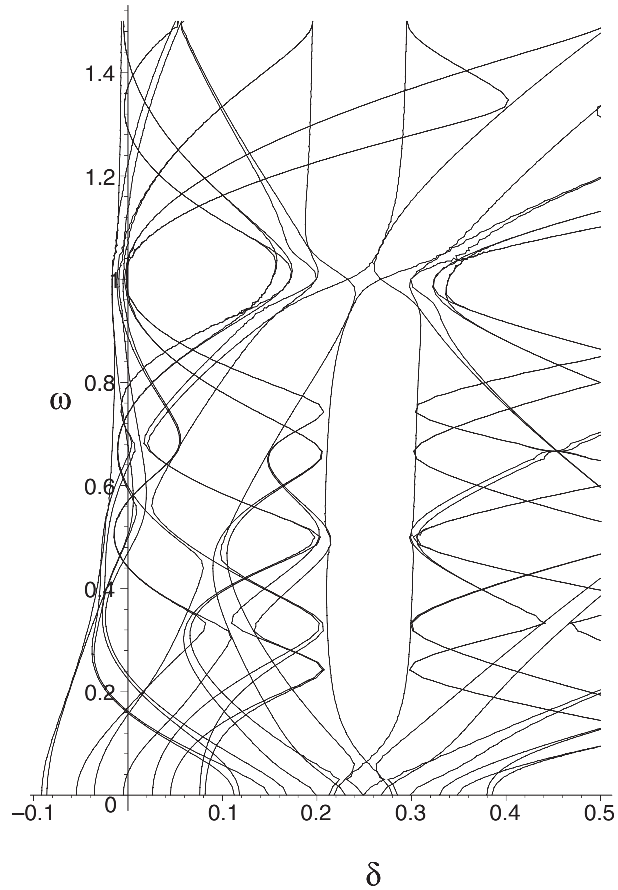

FIG. 1. Stability chart of the quasi-periodic Mathieu equation in the plane (δ,ω) for $N=4$, $\epsilon=0.1$, and $\mu=0$.

$$a_{55}=\frac{\left(\delta-\frac{(1+\omega)^2}{4}\right)^2+\mu^2(1+\omega)^2}{\delta-\frac{(1+\omega)^2}{4}}.$$

The system (13) will have a nontrivial solution only if its determinant vanishes. For each N, the dimension of this system is $2N^2+2N+1$. For the case $N=4$ considered in the current paper, the corresponding system dimension is equal to 41. Nevertheless, the analysis is facilitated by putting the system in upper triangular form. We show in Fig. 1 in the plane (δ,ω) when $\epsilon=0.1$ and $\mu=0$, the stability chart as obtained by Rand *et al.* [26,27]. In this analysis, the vanishing determinant can be given formally in the form $F(\mathrm{Ra},q,\mathrm{Pr},\mathrm{Fr}_1,\Omega_1,\omega)=0$ in which all parameters of the physical problem are taken into account.

The marginal stability curves $\mathrm{Ra}(q)$ are determined numerically by fixing the dimensionless frequency Ω_1, the frequency ratio ω, the Prandtl number Pr, and the Froude number Fr_1. Hereafter, we focus attention on the curves corresponding to the critical Rayleigh number Ra_c and wave numbers q_c versus the dimensionless frequency Ω_1.

IV. RESULTS AND DISCUSSION

Figure 2 illustrates the results of the case where the fluid layer is heated from below and for values of frequencies ratio

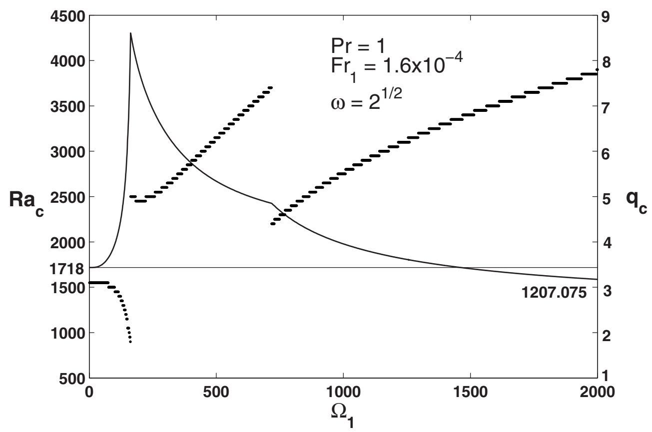

FIG. 2. Heating from below—evolution of the critical Rayleigh number Ra_c and wave number q_c as a function of the dimensionless frequency Ω_1 for $\omega=\sqrt{2}$.

$\omega=\sqrt{2}$, Prandtl number Pr=1, and Froude number $Fr_1=1.6\times10^{-4}$. We see that near $\Omega_1\sim0$, the critical Rayleigh and wave numbers tend, respectively, to the values of the unmodulated case, namely, $Ra_c=1718$ and $q_c=3.14$. Here, it turns out that initially the effect of modulation is stabilizing for Ω_1 lower than 1460, and it becomes destabilizing as Ω_1 increases beyond this value. Furthermore, the critical Rayleigh number reaches the asymptotic value $Ra_c=1207$ for high frequencies. The evolution of the critical wave number gives rise to two jump phenomena when crossing $\Omega_1=163$ and $\Omega_1=718$.

Figure 3 shows the evolution of the critical Rayleigh number as a function of Ω_1 for Pr=1, $Fr_1=1.6\times10^{-4}$, and for different values of the irrational ratio of frequencies ω. In contrast to the curves corresponding to $\omega=\sqrt{3}$, $\omega=\sqrt{5}$, $\omega=\sqrt{11}$, and $\omega=\sqrt{37}$ where we are always in the presence of a stabilizing effect, the curves corresponding to $\omega=\frac{1}{\sqrt{37}}$, $\omega=\frac{1}{\sqrt{2}}$, and $\omega=\sqrt{2}$ give rise to either a stabilizing or destabilizing effect. The zone corresponding to a destabilizing effect narrows dramatically as ω increases, and for a fixed frequency Ω_1, the critical Rayleigh number Ra_c increases with increasing ω. Also, at high frequencies, the asymptotic values of the critical Rayleigh number increases with increasing ω. These results suggest that the onset of convection is well controlled by varying the ratio of frequencies. Note that for values of $\Omega_1\leq200$ approximately, ω has no significant effect on the variation of the critical Rayleigh number.

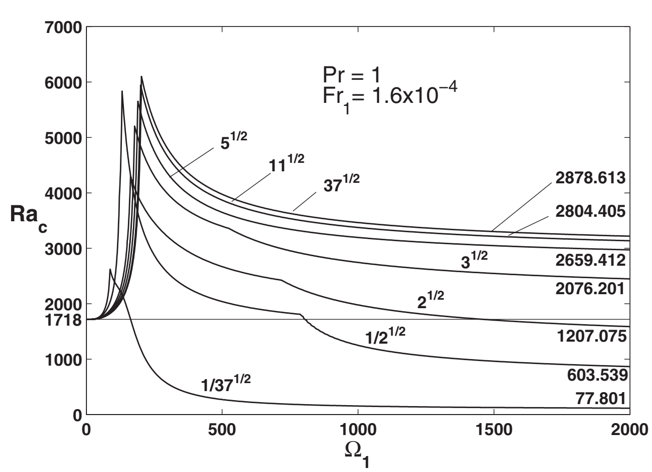

FIG. 3. Heating from below—evolution of the critical Rayleigh number Ra_c as a function of the nondimensional frequency Ω_1 for different values of the irrational frequencies ratio ω.

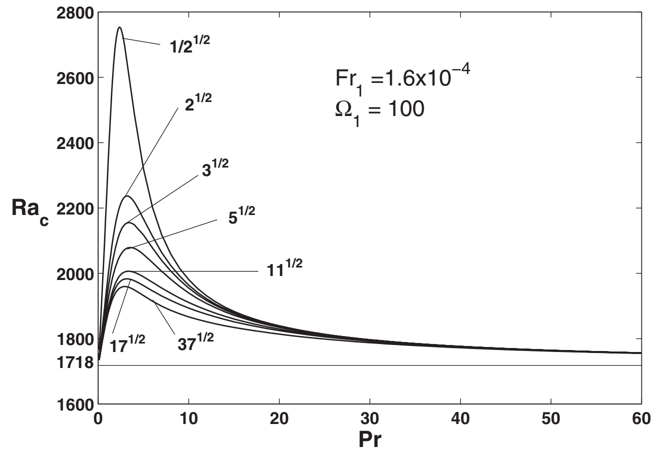

FIG. 4. Heating from below—evolution of the critical Rayleigh number Ra_c as a function of the Prandtl number Pr for $\Omega_1=100$ and for different values of the irrational frequencies ratio ω.

We illustrate in Fig. 4 the dependence of the critical Rayleigh number Ra_c, on the Prandtl number Pr, for $\Omega_1=100$, $Fr_1=1.6\times10^{-4}$, and for different values of the irrational frequencies ratio. It can be seen from Fig. 4 that the largest critical Rayleigh number, corresponding to the maximum of stabilization, increases with decreasing ω and then the stabilizing effect decreases with the frequency ratio. The Prandtl number corresponding to the largest value of Ra_c increases weakly from Pr=2.9 for $\omega=\sqrt{37}$ to Pr=3.5 for $\omega=\sqrt{5}$ and decreases to the value Pr=2.4 for $\omega=\sqrt{2}$. However, for high values of Prandtl number, the critical Rayleigh number for all the profiles tends, as expected, to the value of the unmodulated case $Ra_c=1718$. Indeed, in this situation the inertial term $Pr^{-1}\frac{\partial}{\partial t}$ disappears from Eq. (8).

In Fig. 5, we present the results corresponding to the case

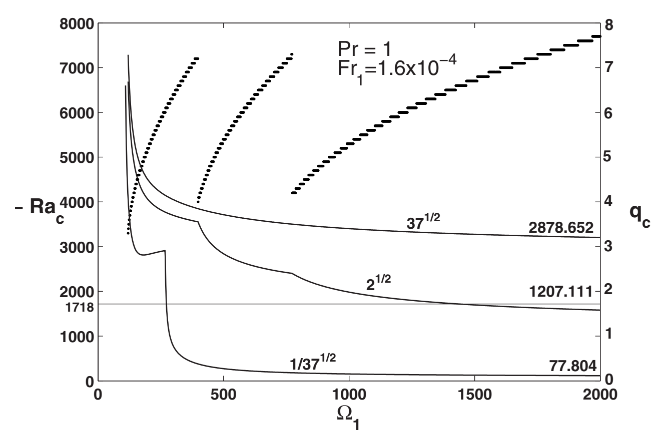

FIG. 5. Heating from above—evolution of the critical Rayleigh number Ra_c and wave number q_c as a function of the dimensionless frequency Ω_1.

of a fluid layer heated from above for $\mathrm{Pr}=1$, $\mathrm{Fr}_1=1.6\times10^{-4}$ and for the values $\omega=1/\sqrt{37}$, $\omega=\sqrt{2}$, and $\omega=\sqrt{37}$. This stable configuration is potentially unstable at high frequencies and stable at low frequencies. Indeed, for each frequency ratio ω, as Ω_1 tends to zero, the critical Rayleigh number increases to high values (stable equilibrium configuration) and decreases with increasing Ω_1 to reach an asymptotic value. We notice that, as in the case of heating from below, the asymptotic critical Rayleigh number decreases with decreasing ω. The evolution of the critical wave number q_c for $\omega=\sqrt{2}$ gives rise to two jump phenomena when crossing $\Omega_1=399$ and $\Omega_2=774$.

V. CONCLUSION

In this work we have studied the effect of vertical quasiperiodic oscillations on the onset of convection in an infinite horizontal layer with rigid boundaries. We have considered the case of a heating from below and the case of a heating from above. The linear equations of convection are reduced to a damped quasiperiodic Mathieu equation where the quasiperiodic solutions characterize the onset of convection. Furthermore, the effect of the frequencies ratio $\omega=\Omega_2/\Omega_1=\omega_2/\omega_1$, on the convection threshold has been observed. It was shown that the modulation with two incommensurate frequencies produces a stabilizing or a destabilizing effect depending on the ratio of the frequencies. This ratio plays an important role in controlling the onset of convection. The effect of the Prandtl number is also studied for $\Omega_1=100$ and it turned out that the stabilizing effect of gravitational quasiperiodic modulation depends strongly on ω for moderate Prandtl numbers.

[1] P. M. Gresho and R. L. Sani, J. Fluid Mech. **40**, 783 (1970).

[2] G. Z. Gershuni and E. M. Zhukhovitskii, *The Convective Stability of Incompressible Fluids* (Keter Publications, Jerusalem, 1976).

[3] M. Lücke and F. Schank, Phys. Rev. Lett. **54**, 1465 (1985).

[4] M. Wadih and B. Roux, J. Fluid Mech. **193**, 391 (1988).

[5] S. Biringen and L. J. Peltier, Phys. Fluids A **2**, 754 (1990).

[6] R. Clever, G. Schubert, and F. H. Busse, J. Fluid Mech. **253**, 663 (1993).

[7] R. Clever, G. Schubert, and F. H. Busse, Phys. Fluids A **5**, 2430 (1993).

[8] U. E. Volmar and H. W. Muller, Phys. Rev. E **56**, 5423 (1997).

[9] J. L. Rogers, M. F. Schatz, J. L. Bougie, and J. B. Swift, Phys. Rev. Lett. **84**, 87 (2000).

[10] S. Aniss, M. Souhar, and M. Belhaq, Phys. Fluids **12**, 262 (2000).

[11] S. Aniss, M. Souhar, and M. Belhaq, C. R. Acad. Sci., Ser. IIb: Mec., Phys., Chim., Astron. **334**, 205 (2000).

[12] S. Aniss, M. Belhaq, and M. Souhar, ASME J. Heat Transfer **123**, 428 (2001).

[13] B. S. Bhadauria, P. K. Bhatia, and L. Debnath, Int. J. Non-Linear Mech. **40**, 475 (2005).

[14] G. Venezian, J. Fluid Mech. **35**, 243 (1969).

[15] S. Rosenblat and D. M. Herbert, J. Fluid Mech. **43**, 385 (1970).

[16] S. Rosenblat and G.A. Tanaka, Phys. Fluids **14**, 1319 (1971).

[17] C. S. Yih and C. H. Li, J. Fluid Mech. **54**, 143 (1972).

[18] R. G. Finucane and R. E. Kelly, Int. J. Heat Mass Transfer **19**, 71 (1976).

[19] M. N. Roppo, S. H. Davis, and S. Rosenblat, Phys. Fluids **27**, 796 (1984).

[20] G. Ahlers, P. C. Hohenberg, and M. Lücke, Phys. Rev. Lett. **53**, 48 (1984).

[21] P. K. Bhatia and B. S. Bhadauria, Z. Naturforsch., A: Phys. Sci. **55**, 957 (2000).

[22] P. K. Bhatia and B. S. Bhadauria, Z. Naturforsch., A: Phys. Sci. **56**, 509 (2001).

[23] B. S. Bhadauria and P. K. Bhatia, Phys. Scr. **66**, 59 (2002).

[24] S. Aniss, M. Belhaq, M. Souhar, and M. G. Velarde, Phys. Scr. **71**, 395 (2005).

[25] O. Bouchra, M. T. Ouazzani and S. Aniss, C. R. Mec. **334**, 205 (2006).

[26] R. Rand, R. R. Zounes, and R. Hastings *Dynamics of a Quasiperiodically Forced Mathieu Oscillator*, Chap. 9 in Nonlinear Dynamics: The Richard Rand 50th Anniversary Volume, edited by A. Guran (World Scientific, 1997)

[27] R. S. Zounes and R. Rand, SIAM J. Appl. Math. **58**, 1094 (1998).

[28] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Oxford University Press, London, 1961).