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Influence of quasi-periodic gravitational modulation on convective instability of reaction fronts in porous media

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ABSTRACT

The influence of a time-dependent gravity on the convective instability of reaction fronts in porous media is investigated in this paper. It is assumed that the time-dependent modulation is quasi-periodic with two frequencies σ_1 and σ_2 that are incommensurate with each other. The model consists of the heat equation, the equation for the depth of conversion and the equations of motion under the Darcy law. The convective threshold is approximated performing a linear stability analysis on a reduced singular perturbation problem using the matched asymptotic expansion method. The reduced interface problem is solved using numerical simulations. It is shown that if the reacting fluid is heated from below, a stabilizing effect of a reaction fronts in a porous medium can be gained for appropriate values of amplitudes and frequencies ratio $\sigma = \frac{\sigma_2}{\sigma_1}$ of the quasi-periodic vibration.

1. Introduction

Several works have been devoted to studying the effect of a periodic vibration on the convective instability of reaction fronts. The case of reaction fronts with liquid reactant and solid product was considered in [1], while the case where the reactant and the product are liquids was analyzed in [2,3]. It was shown that a periodic vibration can affect the onset of convection. Specifically it was indicated that the case where the polymerization front in liquids is different from the case when the polymer is solid. The difference is that in liquids the convective instability may exist also in descending fronts [4]. In the case of reaction fronts in porous media [5], the influence of periodic vibration on convective instability has also been considered. The linear stability analysis along with a direct numerical simulations were performed and the influence of vibration parameters on the onset of convection was examined. It is worth noticing that the problem of convective instability under the influence of periodic gravity or periodic heating of a liquid layer or the effect of periodic magnetic field on magnetic liquid layer has been widely analyzed during the last decades; see for instance [6–16] and references therein.

To the best of our knowledge, while the influence of a periodic modulation on the convective instability was extensively studied, only few works have been devoted to study the effect of a quasi-periodic (QP) vibration on the convective instability. Boulal et al. [17] investigated the effect of a QP gravitational modulation with two incommensurate frequencies on the stability of a heated fluid layer. The threshold of convective instability was determined in the case of heating from below or from above, and it was shown that the frequencies ratio of QP vibration strongly influences the convective instability threshold. A similar study was performed to investigate the influence of QP gravitational modulation on convective instability in Hele-Shaw cell [18]. Moreover, thermal instability in a horizontal Newtonian magnetic liquid layer with

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non-magnetic rigid boundaries was also studied in the presence of a vertical magnetic field and a QP modulation [19]. It was shown that in the case of a heating from below, a QP modulation produces a stabilizing or a destabilizing effect depending on the frequencies ratio. In these works [17–19], the original problem is systematically reduced to a QP Mathieu equation using Galerkin method truncated to the first order. Since the Floquet theory cannot be applied in the QP forcing case, the approach used to obtain the marginal stability curves was based on the application of the harmonic balance method and Hill's determinants [20,21].

The aim of this paper is to investigate the influence of QP gravitational vibration on convective instabilities of reaction fronts in porous media. This study is motivated by applications arising in some physical problems, as for instance, frontal polymerization [22] or the environmental pollution [23] when subjected to a QP vibration. Such a QP vibration may eventually result from a simultaneous existence of a basic vibration applied to the system with a frequency v_1 and of an additional residual vibration having a frequency v_2, such that v_1 and v_2 are incommensurate. Indeed, this residual vibration may come from various sources as machinery, friction or just a modulation of the amplitude of the basic vibration.

In what follows, we consider a QP vibration with two incommensurate frequencies in the vertical direction upon the system containing a reaction reactant and a reaction product. This excitation causes a QP acceleration, b, perpendicular to the reactant-product interface. The time dependence of the instantaneous QP acceleration acting on the fluids is then given by $g + b(t)$, where g is the gravity acceleration and $b(t) = \lambda_1 sin(v_1 t) + \lambda_2 sin(v_2 t)$ where λ_1, λ_2 and v_1, v_2 are the amplitudes and the frequencies of the QP vibration, respectively. Here, we consider reaction fronts in a porous medium with the fluid motion described by the Darcy law and the Boussinesq approximation, which takes into account the temperature dependence of the density only in the volumetric forces.

It is worthy to notice that the problem of reducing the original Navier–Stokes equations to a standard QP Mathieu equation using Galerkin method, harmonic balance method and Hill's determinants [17–19] cannot be exploited here due to the coupling of the concentration and the heat equations (reaction–diffusion problem coupled with the Darcy equation).

Therefore, to obtain the convective stability boundary, we first reduce the original reaction–diffusion problem to a singular perturbation one using the so-called matched asymptotic expansion, we perform a linear stability analysis, and then solve the reduced interface problem using numerical simulations.

The paper is organized as follows. The next section introduces the model, while Section 3 deals with the linear stability by approximating the infinite narrow reaction zone based on the formulation of the interface problem. Results and discussions are also provided in this section. Section 4 concludes the work.

2. Governing equations

We consider an upward propagating reaction front in a porous medium filled by an incompressible reacting fluid submitted to a QP gravitational vibration, as shown in Fig. 1. The model of a such process can be described by a reaction–diffusion system coupled with the hydrodynamics equations under the Darcy law:

$$\frac{\partial T}{\partial t} + \mathbf{v}.\nabla T = \kappa \Delta T + qK(T)\phi(\alpha), \tag{2.1}$$

$$\frac{\partial \alpha}{\partial t} + \mathbf{v}.\nabla \alpha = d\Delta \alpha + K(T)\phi(\alpha), \tag{2.2}$$

$$\mathbf{v} + \frac{K}{\mu}\nabla p = \frac{g\beta K}{\mu}\rho(T - T_0)(1 + \lambda_1 sin(v_1 t) + \lambda_2 sin(v_2 t))\gamma, \tag{2.3}$$

$$\nabla.\mathbf{v} = 0. \tag{2.4}$$

with the following boundary conditions:

$$T = T_i, \alpha = 1 \text{ and } v = 0 \text{ when } y \to +\infty, \tag{2.5}$$

$$T = T_b, \alpha = 0 \text{ and } v = 0 \text{ when } y \to -\infty. \tag{2.6}$$

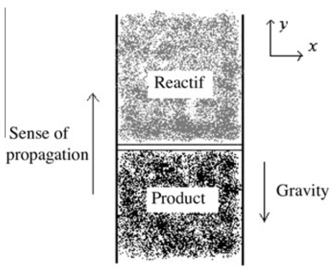

Fig. 1. Sketch of the reaction front propagation.

Here T is the temperature, α the depth of conversion, $\mathbf{v} = (v_x, v_y)$ the fluid velocity, p the pressure, κ the coefficient of thermal diffusivity, d the diffusion, q the adiabatic heat release, g the gravity acceleration, ρ the density, β the coefficient of thermal expansion, μ the viscosity and γ is the unit vector in the upward direction. In addition, T_0 is the mean value of temperature, T_i is an initial temperature while T_b is the temperature of the burned mixture given by $T_b = T_i + q$. The function $K(T)\phi(\alpha)$ is the reaction rate where the temperature dependence is given by the Arrhenius law [24]:

$$K(T) = k_0 \exp\left(-\frac{E}{R_0 T}\right). \tag{2.7}$$

where E is the activation energy, R_0 the universal gas constant and k_0 is the pre-exponential factor. For the asymptotic analysis of this problem we assume that the activation energy is large and we consider zero order reaction for which

$$\phi(\alpha) = \begin{cases} 1 & \text{if } \alpha < 1, \\ 0 & \text{if } \alpha = 1. \end{cases} \tag{2.8}$$

In order to obtain the dimensionless model, we now introduce the spatial variables $x' = \frac{xc_1}{\kappa}$, $y' = \frac{yc_1}{\kappa}$, time $t' = \frac{tc_1^2}{\kappa d}$, velocity $\frac{\mathbf{v}}{c_1}$, pressure $\frac{p\kappa\mu}{K}$ with $c_1 = c/\sqrt{2}$ and frequencies $\sigma_1 = \frac{\kappa}{c_1^2}\nu_1, \sigma_2 = \frac{\kappa}{c_1^2}\nu_2$. Denoting $\theta = \frac{T - T_b}{q}$ and keeping, for convenience, the same notation for the other variables, we obtain the system

$$\frac{\partial\theta}{\partial t} + v\nabla\theta = \Delta\theta + W_Z(\theta)\phi(\alpha), \tag{2.9}$$

$$\frac{\partial\alpha}{\partial t} + v\nabla\alpha = \Lambda\Delta\alpha + W_Z(\theta)\phi(\alpha), \tag{2.10}$$

$$v + \nabla p = R_p(\theta + \theta_0)\begin{pmatrix} 0 \\ 1 \end{pmatrix}(1 + \lambda_1 sin(\sigma_1 t) + \lambda_2 sin(\sigma_2 t)), \tag{2.11}$$

$$div(v) = 0 \tag{2.12}$$

with the following conditions at infinity:

$$\theta = -1, \quad \alpha = 0 \text{ and } v = 0 \text{ when } y \to +\infty, \tag{2.13}$$

$$\theta = 0, \quad \alpha = 1 \text{ and } v = 0 \text{ when } y \to -\infty. \tag{2.14}$$

Here $\Lambda = d/\kappa$ is the inverse of the Lewis number, $R_p = \frac{Kc_1^2 P^2 R}{\mu^2}$, where R is the Rayleigh number and P the Prandtl number defined, respectively, by $R = \frac{g\beta q\kappa^2}{\mu c_1^3}$ and $P = \frac{\mu}{\kappa}$. In addition, we use the parameters $\delta = \frac{R_0 T_b}{E}$ and $\theta_0 = \frac{T_b - T_0}{q}$. The reaction rate is then given by:

$$W_Z(\theta) = Z\exp\left(\frac{\theta}{Z^{-1} + \delta\theta}\right), \tag{2.15}$$

where $Z = \frac{qE}{R_0 T_b^2}$ stands for Zeldovich number.

The linear stability analysis will be carried out in the case of zero Lewis number ($\Lambda = 0$) corresponding to a liquid mixture.

3. Linear stability analysis

3.1. Approximation of infinitely narrow reaction zone

We perform an analytical treatment by reducing the original problem (2.9)–(2.14) to a singular perturbation one where the reaction zone is supposed to be infinitely narrow and the reaction term is neglected outside the reaction zone. This approach, called Zeldovich–Frank–Kamenetskii approximation, is a well-known approach for studying the reaction front propagation [24,25] a closed interface problem is obtained applying a formal asymptotic analysis assuming $\epsilon = \frac{1}{Z}$ is a small parameter. Denoting by $\zeta(t,x)$ the location of the reaction zone in the laboratory frame reference, the new independent variable in the direction of the front propagation is given by

$$y_1 = y - \zeta(t,x). \tag{3.1}$$

Therefore, the new functions θ_1, α_1, $\mathbf{v}_1$, p_1 can be introduced such that:

$$\begin{aligned} &\theta(t,x,y) = \theta_1(t,x,y_1), \alpha(t,x,y) = \alpha_1(t,x,y_1), \\ &\mathbf{v}(t,x,y) = \mathbf{v}_1(t,x,y_1), p(t,x,y) = p_1(t,x,y_1). \end{aligned} \tag{3.2}$$

Consequently, the original Eqs. (2.9)–(2.14) can be re-written in the form (the index 1 for the independent variables is omitted):

$$\frac{\partial\theta}{\partial t}-\frac{\partial\theta}{\partial y_1}\frac{\partial\zeta}{\partial t}+\mathbf{v}.\widetilde{\nabla}\theta=\widetilde{\Delta}\theta+W_Z(\theta)\phi(\alpha), \tag{3.3}$$

$$\frac{\partial\alpha}{\partial t}-\frac{\partial\alpha}{\partial y_1}\frac{\partial\zeta}{\partial t}+\mathbf{v}.\widetilde{\nabla}\alpha=W_Z(\theta)\phi(\alpha), \tag{3.4}$$

$$\mathbf{v}+\widetilde{\nabla}p=R_p(\theta+\theta_0)(1+\lambda_1 sin(\sigma_1 t)+\lambda_2 sin(\sigma_2 t))\gamma, \tag{3.5}$$

$$\frac{\partial v_x}{\partial x}-\frac{\partial v_x}{\partial y_1}\frac{\partial\zeta}{\partial x}+\frac{\partial v_y}{\partial y_1}=0, \tag{3.6}$$

where we have set

$$\widetilde{\Delta}=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y_1^2}-2\frac{\partial\zeta}{\partial x}\frac{\partial^2}{\partial x\partial y_1}+\left(\frac{\partial\zeta}{\partial x}\right)^2\frac{\partial^2}{\partial y_1^2}-\frac{\partial^2\zeta}{\partial x^2}\frac{\partial}{\partial y_1}, \tag{3.7}$$

$$\widetilde{\nabla}=\left(\frac{\partial}{\partial x}-\frac{\partial\zeta}{\partial x}\frac{\partial}{\partial y_1},\frac{\partial}{\partial y_1}\right). \tag{3.8}$$

To approximate the jump conditions and then resolve the interface problem, we apply the matched asymptotic expansions. To this end, the outer solution to the problem is sought in the form

$$\begin{aligned}&\theta=\theta^0+\epsilon\theta^1+\cdots,\quad \alpha=\alpha^0+\epsilon\alpha^1+\cdots,\\&\mathbf{v}=\mathbf{v}^0+\epsilon\mathbf{v}^1+\cdots,\quad p=p^0+\epsilon p^1+\cdots\end{aligned} \tag{3.9}$$

where $(\theta^0,\alpha^0,\mathbf{v}^0)$ is a dimensionless form of the basic solution.

For the inner solution, we introduce the stretching coordinate $\eta = y_1/\epsilon$ and then the inner solution is sought in the form

$$\begin{aligned}&\theta=\epsilon\tilde{\theta}^1+\cdots,\quad \alpha=\tilde{\alpha}^0+\epsilon\tilde{\alpha}^1+\cdots,\\&\mathbf{v}=\tilde{\mathbf{v}}^0+\epsilon\tilde{\mathbf{v}}^1+\ldots,p=\tilde{p}^0+\epsilon\tilde{p}^1+\cdots,\quad \zeta=\tilde{\zeta}^0+\epsilon\tilde{\zeta}^1+\cdots\end{aligned} \tag{3.10}$$

Substituting these expansions, into (3.3)–(3.6), we obtain the following first-order inner problem:

$$\left(1+\left(\frac{\partial\tilde{\zeta}^0}{\partial x}\right)^2\right)\frac{\partial^2\tilde{\theta}^1}{\partial\eta^2}+\exp\left(\frac{\tilde{\theta}^1}{1+\delta\tilde{\theta}^1}\right)\phi(\tilde{\alpha}^0)=0, \tag{3.11}$$

$$-\frac{\partial\tilde{\alpha}^0}{\partial\eta}\frac{\partial\tilde{\zeta}^0}{\partial\eta}-\frac{\partial\tilde{\alpha}^0}{\partial\eta}\left(\tilde{v}_x^0\frac{\partial\tilde{\zeta}^0}{\partial x}-\tilde{v}_y^0\right)=\exp\left(\frac{\tilde{\theta}^1}{1+\delta\tilde{\theta}^1}\right)\phi(\tilde{\alpha}^0), \tag{3.12}$$

$$\frac{\partial\tilde{p}^0}{\partial\eta}=0, \tag{3.13}$$

$$\tilde{v}_x^0+\frac{\partial\tilde{p}^0}{\partial x}-\frac{\partial\tilde{\zeta}^0}{\partial t}\frac{\partial\tilde{p}^1}{\partial\eta}=0, \tag{3.14}$$

$$\tilde{v}_y^0+\frac{\partial\tilde{p}^1}{\partial\eta}=-R_p\theta_0(1+\lambda_1 sin(\sigma_1 t)+\lambda_2 sin(\sigma_2 t)), \tag{3.15}$$

$$-\frac{\partial\tilde{v}_x^0}{\partial\eta}\frac{\partial\tilde{\zeta}^0}{\partial x}+\frac{\partial\tilde{v}_y^0}{\partial\eta}=0. \tag{3.16}$$

On the other hand, the matching conditions are written in the form

$$\eta\to+\infty:\ \tilde{\theta}^1\sim\theta^1\big|_{y_1=0+}+\eta\frac{\partial\theta^0}{\partial y_1}\bigg|_{y_1=0+},\quad \tilde{\alpha}^0\to0,\quad \tilde{\mathbf{v}}^0\to\mathbf{v}^0\big|_{y_1=0+}, \tag{3.17}$$

$$\eta\to-\infty:\ \tilde{\theta}^1\to\theta^1\big|_{y_1=0-},\quad \tilde{\alpha}^0\to1,\quad \tilde{\mathbf{v}}^0\to\mathbf{v}^0\big|_{y_1=0-}. \tag{3.18}$$

We notice that from (3.13) we obtain that $\tilde{p}^0$ does not depend on η, which implies that the pressure is continuous through the interface. Next, denoting by s the quantity

$$s=\tilde{v}_x^0\frac{\partial\tilde{\zeta}^0}{\partial x}-\tilde{v}_y^0, \tag{3.19}$$

we obtain from (3.16) that s does not depend on η. Finally, from 3.14, 3.15 and 3.19 we easily obtain that $\tilde{v}_x^0$ and $\tilde{v}_y^0$ do not depend on η, which provides the continuity of the velocity through the interface.

We next derive the jump conditions for the temperature from (3.11). From (3.12) it follows that $\tilde{\alpha}^0$ is a monotone function and $0<\tilde{\alpha}^0<1$. Since we consider zero-order reaction, we have $\phi(\tilde{\alpha}^0)\equiv1$ and we conclude from (3.11) that $\tilde{\theta}^1$ is also a monotone function. Multiplying (3.11) by $\frac{\partial\tilde{\theta}^1}{\partial\eta}$ and integrating yields

$$\left(\frac{\partial\tilde{\theta}^1}{\partial\eta}\right)^2\Bigg|_{\eta=+\infty} - \left(\frac{\partial\tilde{\theta}^1}{\partial\eta}\right)^2\Bigg|_{\eta=-\infty} = -\frac{2}{A}\int_{-\infty}^{\theta^1}\exp\left(\frac{\tau}{1+\delta\tau}\right)d\tau, \tag{3.20}$$

where we have set

$$A = 1 + \left(\frac{\partial\tilde{\zeta}^0}{\partial x}\right)^2. \tag{3.21}$$

Next, subtracting (3.11) from (3.12) and integrating leads to

$$\frac{\partial\tilde{\theta}^1}{\partial\eta}\bigg|_{\eta=+\infty} - \frac{\partial\tilde{\theta}^1}{\partial\eta}\bigg|_{\eta=-\infty} = -\frac{1}{A}\left(\frac{\partial\tilde{\zeta}^0}{\partial t} + s\right). \tag{3.22}$$

Using the matching conditions and truncating the expansion:

$$\theta^0 \approx \theta, \quad \theta^1|_{y_1=0-} \approx Z\theta|_{y_1=0}\zeta^0 \approx \zeta, \quad \mathbf{v} \approx \mathbf{v}^0, \tag{3.23}$$

the following jump conditions are obtained

$$\left(\frac{\partial\theta}{\partial y_1}\right)^2\Bigg|_{y_1=0+} - \left(\frac{\partial\theta}{\partial y_1}\right)^2\Bigg|_{y_1=0-} = 2Z\left(1 + \left(\frac{\partial\zeta}{\partial x}\right)^2\right)^{-1}\int_{-\infty}^{\theta|_{y_1=0}}\exp\left(\frac{\tau}{Z^{-1}+\delta\tau}\right)d\tau, \tag{3.24}$$

$$\frac{\partial\theta}{\partial y_1}\bigg|_{y_1=0+} - \frac{\partial\theta}{\partial y_1}\bigg|_{y_1=0-} = -\left(1 + \left(\frac{\partial\zeta}{\partial x}\right)^2\right)^{-1}\left(\frac{\partial\zeta}{\partial t} + \left(v_x\frac{\partial\zeta}{\partial x} - v_y\right)|_{y_1=0}\right). \tag{3.25}$$

3.2. Formulation of the interface problem

The interface problem will be written as a system of equations for the reactant and a system of equations for the product as well as the jump conditions.

We have for $y > \zeta$ (in the unburnt medium)

$$\frac{\partial\theta}{\partial t} + \mathbf{v}.\nabla\theta = \Delta\theta, \tag{3.26}$$

$$\alpha \equiv 0, \tag{3.27}$$

$$\mathbf{v} + \nabla p = R_p(\theta+\theta_0)(1+\lambda_1 sin(\sigma_1 t) + \lambda_2 sin(\sigma_2 t))\gamma, \tag{3.28}$$

$$\nabla.\mathbf{v} = 0. \tag{3.29}$$

In the burnt medium ($y < \zeta$), we obtain the system

$$\frac{\partial\theta}{\partial t} + \mathbf{v}.\nabla\theta = \Delta\theta, \tag{3.30}$$

$$\alpha \equiv 1, \tag{3.31}$$

$$\mathbf{v} + \nabla p = R_p(\theta+\theta_0)(1+\lambda_1 sin(\sigma_1 t) + \lambda_2 sin(\sigma_2 t))\gamma, \tag{3.32}$$

$$\nabla.\mathbf{v} = 0. \tag{3.33}$$

We finally complete this system by the following jump conditions at the interface $y = \zeta$

$$[\theta] = 0, \quad \left[\frac{\partial\theta}{\partial y}\right] = \frac{\frac{\partial\zeta}{\partial t}}{1+\left(\frac{\partial\zeta}{\partial x}\right)^2}, \tag{3.34}$$

$$\left[\left(\frac{\partial\theta}{\partial y}\right)^2\right] = -\frac{2Z}{1+\left(\frac{\partial\zeta}{\partial x}\right)^2}\int_{-\infty}^{\theta(\zeta)}\exp\left(\frac{s}{1/Z+\delta s}\right)ds, \tag{3.35}$$

$$[\mathbf{v}] = 0. \tag{3.36}$$

Here we denote by [ ] the quantity $[f] = f|_{\zeta-0} - f|_{\zeta+0}$. The above free boundary problem is completed with the conditions at infinity:

$$y \to +\infty, \theta = -1 \quad \text{and} \quad \mathbf{v} = 0, \tag{3.37}$$

$$y \to -\infty, \theta = 0 \quad \text{and} \quad \mathbf{v} = 0. \tag{3.38}$$

3.3. Travelling wave solution

In this subsection, we perform the linear stability analysis of the steady-state solution for the interface problem. Indeed, this interface problem has a travelling wave solution:

$$\theta(t,x,y) = \theta_s(y-ut), \alpha(t,x,y) = \alpha_s(y-ut) \quad \text{and} \quad \mathbf{v} = 0, \tag{3.39}$$

in which the steady-state solution θ_s is given by:

$$\theta_s(t,y) = \begin{cases} 0 & \text{if } y < 0, \\ e^{-uy} - 1 & \text{if } y > 0, \end{cases} \tag{3.40}$$

and the steady-state solution α_s is written as:

$$\alpha_s(t,y) = \begin{cases} 1 & \text{if } y < 0, \\ 0 & \text{if } y > 0, \end{cases} \tag{3.41}$$

where the number u stands for the stationary front velocity.

Introducing the coordinates in the moving frame defined by $y_1 = y - ut$, the above travelling wave is now considered as a stationary solution of the following problem:

$$\frac{\partial\theta}{\partial t} + u\frac{\partial\theta}{\partial y} + \mathbf{v}.\nabla\theta = \Delta\theta, \tag{3.42}$$

$$\mathbf{v} + \nabla p = R_p(\theta+\theta_0)(1+\lambda_1 sin(\sigma_1 t) + \lambda_2 sin(\sigma_2 t))\gamma, \tag{3.43}$$

$$\nabla.\mathbf{v} = 0, \tag{3.44}$$

together with the jump conditions (3.34)–(3.36).

To perform the linear stability analysis, we introduce a small perturbation to the stationary solution. To this end, we consider a perturbation of the reaction front of the form

$$\zeta(t,x) = ut + \xi(t,x), \quad \text{with } \xi(t,x) = \epsilon_1(t)e^{ikx}. \tag{3.45}$$

The stability of the solution is carried out by assuming the solution of the problem in the perturbed form:

$$\theta = \theta_s + \tilde{\theta}, \qquad v = v_s + \tilde{v}, \tag{3.46}$$

where

$$\begin{aligned} \tilde{\theta}(t,x,y) &= \theta_j(y,t)e^{ikx}, \quad \text{for} \quad j = 1,2, \\ \tilde{v}(t,x,y) &= v_j(y,t)e^{ikx}, \quad \text{for} \quad j = 1,2. \end{aligned} \tag{3.47}$$

Here the index $j = 1$ corresponds to solutions for $z < 0$ and $j = 2$ corresponds to those for $z > 0$. For simplicity, we eliminate the pressure p and the component v_x of the velocity from the interface problem applying two times the operator *curl*. Thus, we obtain the following problem

For the burnt media ($y < 0$):

$$v_1'' - k^2 v_1 = -R_p k^2(1+\lambda_1 sin(\sigma_1 t) + \lambda_2 sin(\sigma_2 t))\theta_1, \tag{3.48}$$

$$\frac{\partial\theta_1}{\partial t} - \theta_1'' - u\theta_1' + k^2\theta_1 = 0. \tag{3.49}$$

For the unburnt media ($y > 0$):

$$v_2'' - k^2 v_2 = -R_p k^2(1+\lambda_1 sin(\sigma_1 t) + \lambda_2 sin(\sigma_2 t))\theta_2, \tag{3.50}$$

$$\frac{\partial\theta_2}{\partial t} - \theta_2'' - u\theta_2' + k^2\theta_2 = u\exp(-uy)v_2, \tag{3.51}$$

Taking into account that

$$\theta|_{\xi=\pm 0} = \theta_s(\pm 0) + \xi\theta_s'(\pm 0) + \tilde{\theta}(\pm 0), \tag{3.52}$$

and

$$\left.\frac{\partial\theta}{\partial y}\right|_{\xi=\pm 0} = \theta_s'(\pm 0) + \xi\theta_s''(\pm 0) + \frac{\partial\tilde{\theta}}{\partial y}(\pm 0), \tag{3.53}$$

we obtain the following jump conditions:

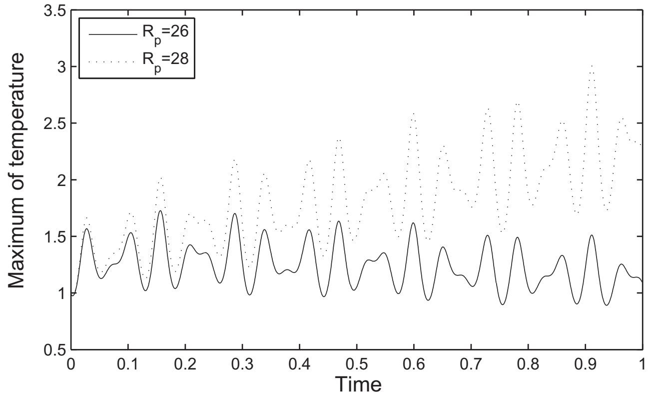

Fig. 2. Maximum of temperature as function of time for $\lambda_1 = 2$, $\lambda_2 = 2$, $\sigma_1 = 100$ and $\sigma_2 = \sqrt{2}\sigma_1$.

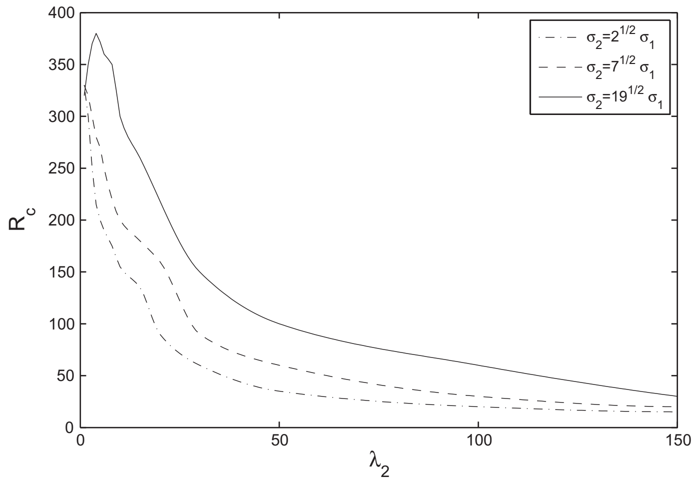

Fig. 3. Critical Rayleigh number as function of the amplitude λ_2 for $\lambda_1 = 5$ and $\sigma_1 = 500$.

$$\theta_2(0,t) - \theta_1(0,t) = u\epsilon_1(t), \tag{3.54}$$

$$\theta_2'(0,t) - \theta_1'(0,t) = -\epsilon_1(t)u^2 - \epsilon_1'(t) + v_1(0,t), \tag{3.55}$$

$$\epsilon_1(t)u^2 + \theta_2'(0,t) = -\frac{Z}{u}\theta_1(0,t), \tag{3.56}$$

$$v_2^{(i)}(0,t) = v_1^{(i)}(0,t) \quad i = 0, 1. \tag{3.57}$$

3.4. Results and discussion

To construct the convective instability boundaries, we solve numerically the problem (3.48)–(3.51) with the jump conditions (3.54)–(3.57) for given values of Z and k and for various values of Rayleigh number R_p. The numerical accuracy is controlled by decreasing the time and space steps.

Fig. 2 depicts the variation of the maximum of temperature as function of time. It can be seen from these plots that if the Rayleigh number R_p is less than a critical value R_c, the solution is decreasing in time which corresponds to a stable (bounded) variation of the maximum of temperature. For values of R_p larger than R_c, the maximum of temperature presents unbounded oscillations which corresponds to unstable solutions. To detect this instability, we start our computations with small Rayleigh numbers and then we increase it slowly until the critical value of the Rayleigh number is captured. The figure shows that the maximum of temperature variation is decreasing for $R_p = 26$ and increasing for $R_p = 28$ indicating that the critical Rayleigh number is approximatively located between, i.e. $R_c \approx 27$.

Fig. 3 shows, for $\lambda_1 = 5$ and $\sigma_1 = 500$, the variation of the critical Rayleigh number as function the amplitude λ_2. The plots indicate that for small values of the frequencies ratio $\sigma = \sigma_2/\sigma_1$, as the amplitude λ_2 increases, the critical Rayleigh number

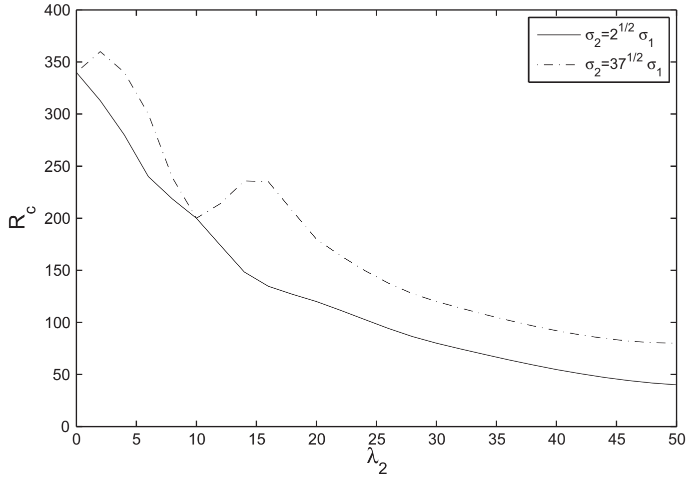

Fig. 4. Critical Rayleigh number as function of the amplitude λ_2 for $\lambda_1 = 5$ and $\sigma_1 = 250$.

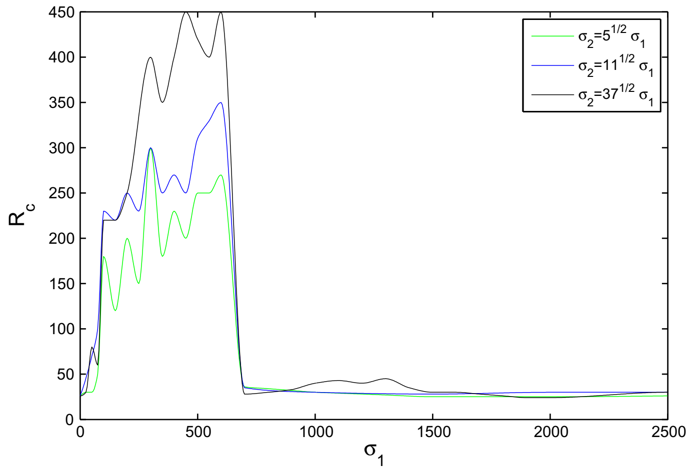

Fig. 5. Critical Rayleigh number as function of frequency σ_1 for $\lambda_1 = \lambda_2 = 5$.

decreases from a certain value of $R_p(\approx 325)$. If σ is increased substantially, a stabilizing effect appears in a region corresponding to small values of λ_2. In this zone, one can expect a regaining of stability of reaction fronts. For higher values of λ_2 the critical Rayleigh number decreases for different σ indicating that large values of λ_2 induce a destabilizing effect. Fig. 4 illustrates similar results for $\sigma_1 = 250$. It is seen in this figure that for higher values of σ, a stabilizing effect appears in two successive regions corresponding, respectively, to small and moderate values of the amplitude λ_2. This result means that increasing σ, stability may be gained in certain specific intervals of λ_2.

Finally, Fig. 5 shows, for given amplitudes and for different values of the frequencies ratio σ, the critical Rayleigh number as function of σ_1. This figure indicates clearly that in the absence of QP vibration ($\sigma_1 = 0, \sigma_2 = 0$), the curves start at the value $R_c = 26$ corresponding to the unmodulated case, which is in good agreement with the previous works [5,26] and hence validating the numerical simulations. This figure also depicts an interesting phenomenon, that is in a certain interval of σ_1, the critical Rayleigh number increases from the unmodulated case $R_c = 26$ while undergoing oscillations. Increasing σ, the oscillating variation of the critical Rayleigh number increases creating a repeated alternating zones where stability is gained. At a certain value of $\sigma_1 \approx 700$, the critical Rayleigh number suddenly drops to reach the unmodulated case, $R_c = 26$. Above $\sigma_1 \approx 700$, the frequencies ratio has no effect on the critical Rayleigh number and the problem becomes equivalent to the unmodulated case.

4. Conclusion

In this work we have studied the effect of a vertical QP gravitational modulation on the convective instability of reaction fronts in porous media. Attention was focused on the case where the QP vibration has two incommensurate frequencies and where the heating is acted from below such that the sense of reaction is opposite to the gravity sense.

To approximate the convective instability threshold, the original reaction–diffusion problem is first reduced to a singular perturbation one using the matched asymptotic expansion. Thus, the linear stability analysis of the steady-state solution for the interface problem is performed. The obtained reduced interface problem is then solved numerically.

It was shown that for relatively small values of the amplitudes λ_1 and λ_2 of the QP vibration, an increase of the frequencies ratio σ has a stabilizing effect (Figs. 3 and 4). The results also revealed that for given values of λ_1 and λ_2 and below a critical value of the frequency σ_1, an increase of the frequencies ratio produces a stabilizing effect. In this interval of σ_1, the convection threshold grows from the critical Rayleigh number of the unmodulated case, $R_c = 26$, undergoing an oscillating variation. This alternating variation indicates that for appropriate values of parameters, a more pronounced stabilizing effect can be gained. At a certain critical value of $\sigma_1 \approx 700$, the critical Rayleigh number suddenly drops and tends back to the unmodulated case, $R_c = 26$. Above $\sigma_1 \approx 700$, the frequencies ratio has no effect on the critical Rayleigh number and the gravity vector can be considered nearly as time-independent.

The results of this work show that in the presence of a QP vibration, the convection instability of reaction fronts in porous media can be controlled and the reaction fronts may remain stable in some regions and for certain combinations of the amplitudes and the frequencies ratio of the QP vibration.

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