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Quasi-Periodic Oscillations, Chaos and Suppression of Chaos in a Nonlinear Oscillator Driven by Parametric and External Excitations

M. BELHAQ and M. HOUSSNI
Laboratory of Mechanics, Group of Nonlinear Oscillations and Chaos, University Hassan II Aïn Chock, Faculty of Sciences, BP 5366 Maârif, Casablanca, Morocco

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Abstract. An analysis is given of the dynamic of a one-degree-of-freedom oscillator with quadratic and cubic nonlinearities subjected to parametric and external excitations having incommensurate frequencies. A new method is given for constructing an asymptotic expansion of the quasi-periodic solutions. The generalized averaging method is first applied to reduce the original quasi-periodically driven system to a periodically driven one. This method can be viewed as an adaptation to quasi-periodic systems of the technique developed by Bogolioubov and Mitropolsky for periodically driven ones. To approximate the periodic solutions of the reduced periodically driven system, corresponding to the quasi-periodic solution of the original one, multiple-scale perturbation is applied in a second step. These periodic solutions are obtained by determining the steady-state response of the resulting autonomous amplitude-phase differential system. To study the onset of the chaotic dynamic of the original system, the Melnikov method is applied to the reduced periodically driven one. We also investigate the possibility of achieving a suitable system for the control of chaos by introducing a third harmonic parametric component into the cubic term of the system.

Keywords: Quasi-periodic excitation, perturbation analysis, generalized averaging, multiple scales, Melnikov technique, suppression of chaos.

1. Introduction

This paper is concerned with the study of the dynamic response of a one-degree-of-freedom system with both quadratic and cubic nonlinearities subjected to combined parametric and external excitation having incommensurate frequencies in the form

$$\ddot{x} + \alpha\dot{x} + \omega_0^2(1 + h\cos(\nu t))x + \beta x^2 + \xi x^3 = \gamma\cos(\omega t), \tag{1}$$

where damping α, nonlinearities β and ξ, parametric excitation amplitude h, frequency ν and external excitation amplitude γ are small. An overdot denotes differentiation with respect to time t; we assume that the frequency ω is close to ω_0. Equation (1) can serve as model of the one-mode vibration of a heavy elastic structure suspended between two fixed supports at the same level and excited by a quasi-periodic forcing. From a physical point of view, the quadratic term may be due to the curvature of the structure, whereas the cubic term may be due to the symmetric nonlinearity of the material. The parametric term may be due to a harmonic axial load.

The behavior of this type of equation with periodic forcing has been investigated extensively by many researchers. For instance, Nayfeh [1] investigated the steady-state (periodic) solutions of Equation (1) with $\gamma = 0$ using a perturbation method. Chaotic behavior has been observed in computerized simulations. Zavodney et al. [2, 3] studied the external driven case ($h = 0$) using the multiple scales method and analyzed the weakly nonlinear dynamic of the system. Chaotic motion was also studied numerically. Szemplinska-Stupnicka et al. [4] considered the parametric excitation case ($\gamma = 0$) and derived approximate criteria for the onset of chaos. Parametrically excited nonlinear oscillators have also been investigated by Nayfeh [5], Lin et al. [6] and others.

Furthermore, the problem of controlling chaotic motion in a periodically forced oscillator has been considered in recent years. Lima and Pettini [7] investigated the control of chaos in the Duffing–Holmes oscillator (Equation (1) with $\beta = 0, h = 0$) with a negative linear stiffness. It has been shown analytically that a small nonlinear resonant parametric perturbation can suppress chaos. This result has been confirmed experimentally by studying a bistable magnetoelastic beam system and introducing parametric periodic perturbation [8]. For an extensive review of methodologies of this topic, see [9].

In contrast to the periodically driven systems, the investigation of quasi-periodically driven oscillators of type (1) has received little attention. Ness [10] considered a system of type (1) with cubic nonlinearity only and performed the averaging method to investigate resonance phenomena and stability. Moon and Holmes [11] reported an experimental study and developed the so-called double Poincaré section to study the dynamics and strange attractors of a quasi-periodically forced mechanical oscillator. HaQuang et al. [12] analyzed the interaction between parametric and external excitations in the particular case $\omega = \nu$ of the system (1) and studied solutions using the multiple scales method and numerical calculations. Szabelski and Warminski [13] considered a self-excited system with parametric and external excitations and analyzed the influence of the self-excitation in the commensurate case $2\omega = \nu$. The analysis stability of a quasi-periodically forced linear Mathieu oscillator was also conducted by Rand et al. [14] using regular perturbation, harmonic balance method and numerical simulations. Yagazaki and co-workers [15, 16] studied the dynamics of a system of type (1) with a cubic nonlinear term. The standard Van der Pol averaging transformation and the second-order averaging method were applied to obtain a reduced periodic system. A linear approximation of periodic solutions of the reduced system and chaotic dynamics was reported.

The aim of this work is twofold. First, we construct an explicit analytical approximate solution of quasi-periodic oscillations and we analyze the system control parameter space for the possible occurrence of chaotic responses by means of the Melnikov method. The second aim concerns the realization of a suitable control system by introducing a third harmonic parametric component into a nonlinear term.

To construct an analytical approximation of the quasi-periodic solutions, a new method requiring two steps is proposed. The first step mainly consists of transforming the original quasi-periodic system (1) to a reduced periodically driven one using the generalized averaging method. To approximate the periodic solutions of the reduced system, corresponding to the quasi-periodic solution of Equation (1), we apply the multiple scales perturbation method in a second step. This technique gives us new autonomous differential system whose fixed points are amplitudes and phases of the previous periodically driven system. The steady-state solutions of this second reduced amplitude-phase system correspond precisely to the quasi-periodic solutions of the initial system (1).

The second purpose of this work is mainly concerned with the suppression of chaotic motions of the system (1). Applying the Melnikov technique [17] to the reduced system with periodic coefficients, we approximate the homoclinic diagram bifurcation. By introducing a third harmonic component (resonant parametric perturbation) into the cubic nonlinear term of the original system (1), we realize a suitable system for control of chaos in the system. More precisely, if the parameters of the system (1) are such that the system is in chaotic state, we are interested in its possible suppression by introducing resonant parametric perturbation involving a third frequency in the system.

In Section 2 we investigate fixed points and quasi-periodic motions of the system (1) including homoclinic orbits of the corresponding Hamiltonian system. The generalized averaging and multiple scales perturbation methods are successively performed to reduce the original quasi-periodic system twice. Section 3 is devoted to the onset of chaotic motions of the first reduced system with periodic coefficients using the Melnikov technique. In Section 4 we study the suppression of chaos of the system (1). Section 5 is devoted to numerical simulations. We conclude in Section 6 with a discussion of our results.

2. Quasi-Periodic Motions

In this section we develop a new strategy to construct an analytical approximation of quasi-periodic solutions of the system (1), combining the generalized averaging and the multiple scales techniques. We start with the application of a modified generalized averaging method to obtain a reduced differential system with periodic coefficients. The multiple scales technique is then performed to approximate the periodic solutions of this reduced system corresponding to the quasi-periodic solution of the original one.

2.1. Generalized Averaging Theory and Reduced System

Let us construct an approximation of the quasi-periodic solutions of Equation (1) close to the external resonance of order q. We then impose $\omega_0^2 = (p\omega/q)^2 + \delta$, where δ is a detuning parameter; p and q are relatively prime. We introduce two small parameters ε and μ, such that $0 < \varepsilon \ll \mu \ll 1$ and let $\alpha = \mu\tilde{\alpha} = \mu\varepsilon^2\tilde{\tilde{\alpha}}$, $\gamma = \varepsilon\tilde{\gamma}$, $\nu = \varepsilon\tilde{\nu}$, $\beta = \varepsilon\tilde{\beta}$, $h = \mu\tilde{h} = \mu\varepsilon^2\tilde{\tilde{h}}$, $\xi = \varepsilon^2\tilde{\tilde{\xi}}$ and $\delta = \varepsilon^2\tilde{\delta}$. Then the system (1) becomes

$$\begin{aligned}\ddot{x} + \left(\frac{p}{q}\,\omega\right)^2 x \;=\; &\varepsilon[-\tilde{\beta}x^2 + \tilde{\gamma}\cos(\omega t)] \\ &+ \varepsilon^2[-\tilde{\delta}x - \mu\tilde{\tilde{\alpha}}\dot{x} - \mu\tilde{\tilde{h}}(p\omega/q)^2\cos(\tilde{\nu}\tau)x - \tilde{\xi}x^3] + \cdots . \qquad (2)\end{aligned}$$

We immediately see that this system (2) contains a ‘fast’ time t and a ‘slow’ time $\tau = \varepsilon t$, which are assumed to be independent.

Using the generalized averaging technique [18], adapted principally in this study to the quasi-periodically driven oscillators [19], the approximation of quasi-periodic solutions of Equation (2) is sought in the form

$$x = a\cos\left(\frac{p}{q}\omega t + \theta\right) + \varepsilon U_1\left(a, \frac{\Psi}{p}, \frac{q}{p}\theta, \tilde{\nu}\tau\right) + \varepsilon^2 U_2\left(a, \frac{\Psi}{p}, \frac{q}{p}\theta, \tilde{\nu}\tau\right) + \cdots, \qquad (3)$$

$$\left\{\begin{aligned}\frac{\mathrm{d}a}{\mathrm{d}t} &= \varepsilon A_1(a,\theta,\tilde{\nu}\tau) + \varepsilon^2 A_2(a,\theta,\tilde{\nu}\tau) + \varepsilon^3 A_3(a,\theta,\tilde{\nu}\tau) + \cdots, \\ \frac{\mathrm{d}\theta}{\mathrm{d}t} &= \varepsilon B_1(a,\theta,\tilde{\nu}\tau) + \varepsilon^2 B_2(a,\theta,\tilde{\nu}\tau) + \varepsilon^3 B_3(a,\theta,\tilde{\nu}\tau) + \cdots,\end{aligned}\right. \qquad (4)$$

where $\Psi = (p/q)\omega t + \theta$, and the unknown coefficients A_i, B_i (2π-periodic in both θ and νt) are determined by the condition that secular terms vanish in the correction functions

U_1, U_2 (2π-periodic in θ, ωt and νt). The amplitude a and the phase θ of the quasi-periodic solution (3) are assumed to vary with time according to Equation (4). To be more general in our analysis, let us retain p and q in the formulation. Thereby, we can obtain an approximation in the vicinity of several resonances simultaneously. Applying the same process as given in [20] to Equation (2), we obtain the following second-order approximation of the solution

$$x = a\cos\left(\frac{p}{q}\,\omega t + \theta\right) + \varepsilon U_1\left(a, \frac{\Psi}{p}, \frac{q}{p}\,\theta, \tilde{\nu}\tau\right),$$

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\frac{q}{2p\omega}\left\{\begin{array}{l} \varepsilon[\delta_{q,p}F^{(1)}_{q,-1}(a,\tilde{\nu}\tau)\sin(q\theta/p)] \\ +\,\varepsilon^2[\mathrm{sg}(q-p)\delta_{q-p,p}F^{(2)}_{q-p,-1}(a,\tilde{\nu}\tau)\sin(q\theta/p) + G^{(2)}_{p,0}(a,\tilde{\nu}\tau)] \end{array}\right\},$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{q}{2pa\omega}\left\{\begin{array}{l} \varepsilon[\delta_{q,p}F^{(1)}_{q,-1}(a,\tilde{\nu}\tau)\cos(q\theta/p)] \\ +\,\varepsilon^2[\mathrm{sg}(q-p)\delta_{q-p,p}F^{(2)}_{q-p,-1}(a,\tilde{\nu}\tau)\cos(q\theta/p) + F^{(2)}_{p,0}(a,\tilde{\nu}\tau)] \end{array}\right\}, \quad (5)$$

where

$$F^{(2)}_{p,0}(a,\tilde{\nu}\tau) = -a\tilde{\delta} - \frac{3}{4}\tilde{\xi}a^3 + \frac{5}{6}\,\frac{q^2\tilde{\beta}^2}{p^2\omega^2}\,a^3 - \mu a\left(\frac{p\omega}{q}\right)^2\tilde{\tilde{h}}\cos(\tilde{\nu}\tau),$$

$$G^{(2)}_{p,0}(a,\tilde{\nu}\tau) = \mu\tilde{\tilde{\alpha}}\,\frac{p\omega}{q}\,a, \quad F^{(1)}_{q,-1}(a,\tilde{\nu}\tau) = \tilde{\gamma}$$

and

$$F^{(2)}_{q-p,-1}(a,\tilde{\nu}\tau) = \frac{(\delta_{q,p}-1)}{(1-(q/p)^2)}\,\frac{q^2\tilde{\beta}\tilde{\gamma}}{p^2\omega^2}\,a.$$

The $\mathrm{sg}(\cdot)$ is the sign function and $\delta_{i,j}$ is the Kronecker symbol. According to the expressions (5), this expansion to second order allows us to study two resonances simultaneously, namely, $p = 1; q = 1$ and $p = 1; q = 2$. Indeed, $\delta_{i,j}$ is different from zero for these values of p and q. Hereafter, we consider the fundamental resonance case: $p = 1; q = 1$. Note that the case $p = 1; q = 2$ can be treated similarly. Hence, the system (5) reduces to the following nonautonomous amplitude-phase system

$$\left\{\begin{array}{l} \frac{\mathrm{d}J}{\mathrm{d}t} = \frac{-\gamma}{2\omega}\sqrt{2J}\sin(\theta) - \mu(\tilde{\alpha}J), \\ \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\delta}{2\omega} + \frac{A_0}{4\omega}J - \frac{\gamma}{2\omega\sqrt{2J}}\cos(\theta) + \mu\left(\frac{\tilde{h}\omega}{2}\cos(\nu t)\right), \end{array}\right. \quad (6)$$

where $a^2 = 2J$ and $A_0 = 3\xi - 10\beta^2/3\omega^2$. The periodic solutions of the periodically driven system (6) correspond to the quasi-periodic solutions of (1). Let us consider the autonomous case $\mu = 0$, which we refer to as the unperturbed system, in the form

$$\left\{\begin{array}{l} \frac{\mathrm{d}J}{\mathrm{d}t} = \frac{-\gamma}{2\omega}\sqrt{2J}\sin\theta, \\ \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\delta}{2\omega} + \frac{A_0}{4\omega}J - \frac{\gamma}{2\omega\sqrt{2J}}\cos\theta. \end{array}\right. \quad (7)$$

The fixed points of these modulation Equations (7), corresponding to $\mathrm{d}J/\mathrm{d}t = \mathrm{d}\theta/\mathrm{d}t = 0$, are the roots of the cubic equation

$$AJ^3 + BJ^2 + CJ + D = 0, \quad (8)$$

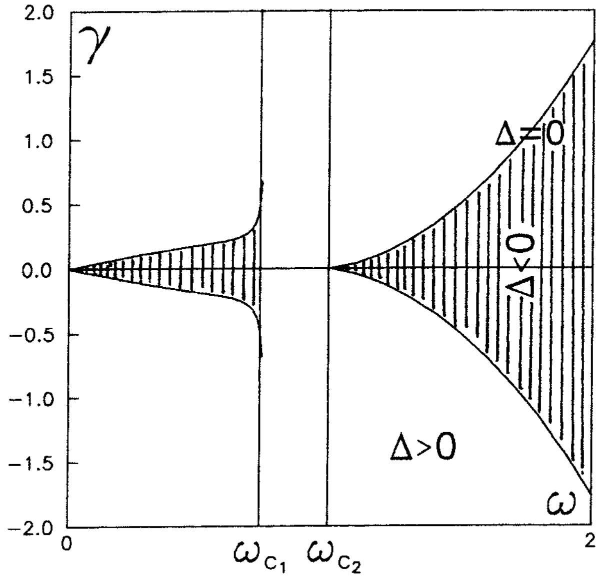

Figure 1. Bifurcation curves of quasi-periodic solutions in the (γ, ω) plane for $\xi = 2$, $\beta = 1$ and $\omega_0 = 1$.

with $A = 1$, $B = 4(\delta/A_0)$, $C = (2\delta/A_0)^2$ and $D = -2(\gamma/A_0)^2$. By writing $J = R - (B/3A)$ in Equation (8) we obtain

$$R^3 + PR + Q = 0, \tag{9}$$

which gives a real solution when one of the following conditions are satisfied (see the corresponding region in the parameter space, Figure 1).

$$\begin{cases} \Delta < 0 \quad \text{or} \quad \Delta = 0 \quad \text{or} \quad \Delta > 0, \\ \delta A_0 < 0. \end{cases} \tag{10}$$

The quantity $\Delta = (P/3)^3 + (Q/2)^2$ is the discriminant of Equation (9) in which $P = (C/A) - (B^2/3A^2)$ and $Q = (2/27)(B/A)^3 - (BC/3A^2) + (D/A)$. Figure 1 illustrates the bifurcation curves of quasi-periodic solutions of Equation (1) for the parameter values $\xi = 2, \beta = 1$ and $\omega_0 = 1$. Three regions can be distinguished corresponding to different frequencies. In both hatched regions, there exist three fixed points, two centers and one saddle, whereas in the unhatched region only one center exists. The Hamiltonian energy related to the unperturbed system (7) is given by

$$H(J, \theta) = \frac{\delta}{2\omega} J + \frac{A_0}{8\omega} J^2 - \frac{\gamma}{2\omega} \sqrt{2J} \cos\theta. \tag{11}$$

We now describe this integrable structure, which will be used in the following sections. Hereafter we consider the case $\delta A_0 < 0$ and $\Delta < 0$. Hence, there exist two centers $(J, \theta) = (j_1, \theta_1)$, (j_3, θ_3) and a hyperbolic saddle $(J, \theta) = (j_2, \theta_2)$ such that $0 < j_1 < j_2 < j_3$. Denote these fixed points by c_1, s_2 and c_3, respectively. The level set which is given by

$$H(J, \theta) = H(j_2, \theta_2) = H_0, \tag{12}$$

is composed of two homoclinic orbits Γ_+, Γ_-, and the fixed point s_2. The homoclinic orbits, $(J_\pm(t), \theta_\pm(t))$, are given by

$$\begin{cases} J_\pm(t) = G_\pm(t) + j_2, \\ \theta_\pm(t) = \arccos\left[\frac{1}{\frac{\gamma}{2\omega}\sqrt{2J_\pm(t)}}\left(\frac{\delta}{2\omega}J_\pm(t) + \frac{A_0}{8\omega}J_\pm^2(t) - H_0\right)\right], \end{cases} \tag{13}$$

where

$$G_\pm(t) = \begin{cases} \pm\frac{2c_+c_-}{(c_+-c_-)\cosh(ct)\pm(c_++c_-)} & \text{if } A_0 > 0, \\ \mp\frac{2c_+c_-}{(c_+-c_-)\cosh(ct)\mp(c_++c_-)} & \text{if } A_0 < 0, \end{cases}$$

$$c = \frac{|A_0|}{8\omega}\sqrt{-c_+c_-}, \quad c_\pm = 2(k \pm \sqrt{2kj_2}) \quad \text{and} \quad k = -2\frac{\delta}{A_0} - j_2.$$

2.2. Periodic Orbits of the Reduced System by Multiple Scales Technique

To determine the periodic solutions of the reduced system (6), it is convenient to transform the polar form (6) using the variable change

$$u = \sqrt{2J}\,\cos\theta, \quad v = -\sqrt{2J}\,\sin\theta, \tag{14}$$

to the Cartesian one

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\delta}{2\omega}v + \frac{A_0}{8\omega}v(u^2+v^2) + \mu\left(-\frac{\tilde{\alpha}}{2}u + \frac{\tilde{h}\omega}{2}v\cos(\nu t)\right), \\ \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\gamma}{2\omega} - \frac{\delta}{2\omega}u - \frac{A_0}{8\omega}u(u^2+v^2) + \mu\left(-\frac{\tilde{\alpha}}{2}v - \frac{\tilde{h}\omega}{2}u\cos(\nu t)\right). \end{cases} \tag{15}$$

To approximate the periodic orbits of the reduced (averaged) system (6), Yagazaki et al. [15] set the periodic solutions $(J_i(t), \theta_i(t))$ with $i = 1, 2, 3$, in the vinicity of the stationary solutions in the form

$$\begin{cases} J_i(t) = j_i + \mu\zeta_i(t), \\ \theta_i(t) = \theta_i + \mu\eta_i(t), \end{cases} \tag{16}$$

where (j_i, θ_i) are the fixed points of the system (7) and $\zeta_i(t)$, $\eta_i(t)$ are perturbative functions. By substituting these last expressions in the reduced system (6), the functions $\zeta_i(t)$ and $\eta_i(t)$ are determined by solving a linear differential system. This gives a simple expression of ζ_i and η_i in the form $\zeta_i(t) = A_i\cos(\nu t)$ and $\eta_i(t) = B_i\sin(\nu t) + C_i$, where A_i, B_i, C_i, are constants depending on the parameters of the system and j_i. In this approach, a linear approximation of the periodic solutions of the reduced system (6) is obtained close to the stationary solutions of Equation (7).

In this section, we propose a new strategy to approximate the periodic solutions of the reduced system (6). More precisely, we directly attack the transformed reduced system (15) by constructing an asymptotic expansion of the periodic solutions close to the stationary solutions using the multiple scales technique [21], realizing a nonlinear approximation of the periodic solutions instead of the linear one [15].

Hence, the second-order uniform approximate solution of the system (15) can be sought in the form

$$\begin{cases} u(t;\mu) = u_0 + \mu u_1(T_0, T_1, T_2) + \mu^2 u_2(T_0, T_1, T_2) + \cdots, \\ v(t;\mu) = v_0 + \mu v_1(T_0, T_1, T_2) + \mu^2 v_2(T_0, T_1, T_2) + \cdots, \end{cases} \tag{17}$$

where $T_n = \mu^n t$ and (u_0, v_0) denotes the fixed point of the unperturbed system (7) given in the Cartesian form (14). In terms of the variables T_n, the time derivative becomes $\mathrm{d}/\mathrm{d}t = \varepsilon D_1 + \varepsilon^2 D_2 + \cdots$, where $D_n = \partial/\partial T_n$.

Substituting Equation (17) into Equation (15) and equating coefficients of same powers of μ, one obtains the following systems

$$\begin{cases} D_0^2 u_1 + \Omega^2 u_1 = -\tilde{f}\cos(\nu t), \\ v_1 = \frac{A_0}{8\omega k}\left(\frac{\tilde{\alpha}}{2}u_0 + D_0 u_1\right), \end{cases} \tag{18a,b}$$

$$\begin{cases} D_0^2 u_2 + \Omega^2 u_2 = -\left(\frac{\tilde{\alpha}}{2}\right)^2 u_0 - \frac{A_0\gamma}{16\omega^2}(3u_1^2 + v_1^2) - \tilde{\alpha} D_0 u_1 - 2D_0 D_1 u_1 \\ \qquad - \frac{\tilde{f}}{u_0} u_1 \cos(\nu t) + \frac{A_0 u_0}{4\omega} D_0(u_1 v_1) + \frac{\tilde{h}\omega}{2} D_0(v_1 \cos(\nu t)), \\ v_2 = \frac{A_0}{8\omega k}\left(D_0 u_2 + D_1 u_1 - \frac{A_0 u_0}{4\omega} u_1 v_1 + \frac{\tilde{\alpha}}{2} u_1 - \frac{\tilde{h}\omega}{2} v_1 \cos(\nu t)\right), \end{cases} \tag{19}$$

$$\begin{aligned} D_0^2 u_3 + \Omega^2 u_3 \;=\; & -\frac{\tilde{\alpha}}{2} D_0 u_2 - D_0 D_1 u_2 - D_0 D_2 u_1 \\ & + \frac{\tilde{h}\omega}{2} D_0(v_2 \cos(\nu t)) + \frac{A_0}{8\omega} D_0[v_1(u_1^2 + v_1^2)] \\ & + \frac{A_0 u_0}{4\omega} D_0(v_1 u_2 + v_2 u_1) - \frac{8\omega k}{A_0}(D_1 v_2 + D_2 v_1) - \frac{4\omega\tilde{\alpha} k}{A_0} v_2 \\ & - \frac{\tilde{f}}{u_0} u_2 \cos(\nu t) - k[u_1(u_1^2 + v_1^2) + 6u_0 u_1 u_2 + 2u_0 v_1 v_2], \end{aligned} \tag{20}$$

where

$$\Omega = \left[\left(\frac{\delta}{2\omega} + \frac{A_0}{8\omega}u_0^2\right)\left(\frac{\delta}{2\omega} + \frac{3A_0}{8\omega}u_0^2\right)\right]^{1/2}$$

is the proper frequency of the system (15),

$$\tilde{f} = \frac{\tilde{h}\omega u_0}{2}\left(\frac{\delta}{2\omega} + \frac{A_0}{8\omega}u_0^2\right)$$

and

$$k = \frac{A_0}{8\omega}\left(\frac{\delta}{2\omega} + \frac{A_0}{8\omega}u_0^2\right).$$

To study the weakly nonlinear fundamental oscillations close to the subharmonic resonances of order m/n and to construct the approximate solutions near these resonances, say $\Omega \cong$

$(m/n)\nu$, we introduce the detuning parameter σ according to $\nu = (n/m)\Omega + \sigma$ and write $\sigma = \mu^2\tilde{\sigma}$ and $\nu T_0 = (n/m)\Omega T_0 + \tilde{\sigma} T_2$. Note that these subharmonic solutions can only be constructed in the vicinity of a center. Then, Equation (18a) becomes

$$D_0^2 u_1 + \Omega^2 u_1 = -\frac{\tilde{f}}{2}\, e^{i\tilde{\sigma} T_2}\, e^{i(n/m)\Omega T_0} + \text{cc}. \tag{21}$$

The solution of this can be expressed in the following complex form

$$u_1 = -\frac{(1-\delta_{n,m})}{2\Omega^2\left(1-\frac{n^2}{m^2}\right)}\,\tilde{f}\, e^{i\tilde{\sigma} T_2} e^{i(n/m)\Omega T_0} + A(T_1, T_2)\, e^{i\Omega T_0} + \text{cc}, \tag{22}$$

where ‘cc’ stands for complex conjugate and $A(T_1, T_2)$ is the slowly varying complex amplitude determined from the higher order expansion.

In this work we focus our attention on the resonance case $n = 2; m = 1$. The analysis of the other resonances can be carried out similarly.

Removal of secular terms at the second order and then at the third order leads to the two following solvability conditions, respectively

$$C_1 + C_{(n/m)-1} = 0, \tag{23}$$

$$Q_1 + Q_{(n/m)-1} = 0, \tag{24}$$

where C_1, $C_{(n/m)-1}$, Q_1 and $Q_{(n/m)-1}$ depend on the parameters A, u_0, Ω, n, m, $\tilde{\sigma}$, $\tilde{\alpha}$, $\tilde{h}$, δ, γ and A_0.

These equations can be combined to describe the modulation of the complex amplitude to the third order with respect to the original time. Combining Equations (23) and (24) in the expression $\mathrm{d}/\mathrm{d}t = \varepsilon D_1 + \varepsilon^2 D_2 + \cdots$ yields

$$\begin{aligned} -2i\Omega\dot{A} \ &+\ i\left[\mu C_1^A A + \mu^2 (Q_1^{\bar{A}e^{i\tilde{\sigma}T_2}} + Q_{(n/m)-1}^{\bar{A}e^{i\tilde{\sigma}T_2}})\bar{A}\, e^{i\tilde{\sigma}T_2}\right] \\ &+\ \mu^2 (Q_1^A + Q_{(n/m)-1}^A + Q_1^{\bar{A}A^2} A\bar{A})A + \mu C_{(n/m)-1}^{\bar{A}e^{i\tilde{\sigma}T_2}} \bar{A} e^{i\tilde{\sigma}T_2} = 0, \end{aligned} \tag{25}$$

where

$$C_1 = -2i\Omega D_1 A + iC_1^A A, \quad C_{(n/m)-1} = C_{(n/m)-1}^{\bar{A}e^{i\tilde{\sigma}T_2}} \bar{A} e^{i\tilde{\sigma}T_2},$$

$$Q_1 = -2i\Omega D_2 A + Q_1^A A + Q_1^{\bar{A}A^2}\, \bar{A}A^2 + i\, Q_1^{\bar{A}e^{i\tilde{\sigma}T_2}}\, \bar{A} e^{i\tilde{\sigma}T_2},$$

$$Q_{(n/m)-1} = i\, Q_{(n/m)-1}^{\bar{A}e^{i\tilde{\sigma}T_2}} \bar{A} e^{i\tilde{\sigma}T_2} + Q_{(n/m)-1}^A A.$$

The expressions C_1^A, $C_{(n/m)-1}^{\bar{A}e^{i\tilde{\sigma}T_2}}$, Q_1^A, $Q_1^{\bar{A}A^2}$, $Q_1^{\bar{A}e^{i\tilde{\sigma}T_2}}$, $Q_{(n/m)-1}^{\bar{A}e^{i\tilde{\sigma}T_2}}$ and $Q_{(n/m)-1}^A$ are given in Appendix 1.

By letting $A = (1/2)a_s e^{i\beta_s}$ where a_s and β_s are real functions, substituting into Equation (25), separating real and imaginary parts and solving, we obtain the second reduced modulation autonomous system

$$\begin{cases} \frac{\mathrm{d}a_s}{\mathrm{d}t} = \frac{E}{2\Omega} a_s + \frac{F}{2\Omega} a_s \cos 2\theta_s + \frac{G}{2\Omega} a_s \sin 2\theta_s, \\ a_s \frac{\mathrm{d}\theta_s}{\mathrm{d}t} = \frac{\sigma\Omega + H}{2\Omega} a_s + \frac{A_\mu}{2\Omega} a_s^3 + \frac{G}{2\Omega} a_s \cos 2\theta_s - \frac{F}{2\Omega} a_s \sin 2\theta_s; \quad \theta_s = \frac{\sigma}{2} t - \beta_s, \end{cases} \tag{26}$$

where E, F, G, H and A_μ are given in Appendix 1. The periodic solutions of Equation (15) or (6) correspond precisely to the stationary solutions of the system (26) given by setting $\mathrm{d}a_s/\mathrm{d}t = 0$ and $\mathrm{d}\theta_s/\mathrm{d}t = 0$. Hence it follows on the one hand that $a_s = 0$ is a possible solution and, on the other hand, combining the above Equations (26) leads to

$$A_s a_s^4 + 2B_s a_s^2 + C_s = 0, \tag{27}$$

where $A_s = A_\mu^2$, $B_s = DA_\mu$, $C_s = D^2 + E^2 - (G^2 + F^2)$ and $D = \sigma\Omega + H$.

Equation (27), governing the steady-state response, gives real and positive solutions in the form $a_{s1,s2}^2 = (-B_s \pm \sqrt{\Delta_s})/A_s$ if one of the following conditions is satisfied

$$\begin{cases} \Delta_s > 0, \\ \frac{C_s}{A_s} > 0 \quad \text{and} \quad \frac{B_s}{A_s} < 0, \end{cases} \qquad \text{or} \qquad \begin{cases} \Delta_s > 0, \\ \frac{C_s}{A_s} < 0, \end{cases} \tag{28}$$

where $\Delta_s = B_s^2 - A_s C_s$ is the discriminant of Equation (27). These conditions are the threshold for the existence of subharmonic oscillations of order one-half. The cycle S_2 corresponding to the stationary solution a_{s2} is a saddle while the cycle F_2 corresponding to a_{s1} is a focus. Next, one can determine the stability of the solution F_2.

Finally, using Equations (7), (14), (17) and the solutions u_n, v_n determined at different orders, the approximation up to second order of the periodic solutions of the reduced system (15) is given by

$$\begin{cases} u(t) = u_0 + \frac{1}{\Omega^2} u_{2,0} + \frac{1}{3\Omega^2} f\cos(\nu t) + \frac{2}{3\Omega^2} u_{2,(n/m)} \sin(\nu t) \\ \quad - \frac{2}{15\Omega^2} u_{2,(2n/m)} \cos(2\nu t) + a\cos\left(\frac{1}{2}\nu t - \theta_s\right) \\ \quad - \frac{1}{6\omega^2} u_{2,2} a^2 \cos\left[2\left(\frac{1}{2}\nu t - \theta_s\right)\right] - \frac{1}{8\Omega^2} u_{2,(n/m)+1} a\cos\left(\frac{3}{2}\nu t - \theta_s\right), \\ v(t) = \frac{A_0 u_0}{16k\omega}\alpha - \frac{A_0}{12k\omega\Omega} f\sin(\nu t) - 2v_{2,(2n/m)} \sin(2\nu t) - \frac{A_0\Omega}{8k\omega} a\sin\left(\frac{1}{2}\nu t - \theta_s\right) \\ \quad + 2v_{2,(n/m)}\cos(\nu t) + v_{2,1} a\cos\left(\frac{1}{2}\nu t - \theta_s\right) + v_{2,1b} a\sin\left(\frac{1}{2}\nu t + \theta_s\right) \\ \quad - \frac{1}{2} v_{2,2} a^2 \sin\left[2\left(\frac{1}{2}\nu t - \theta_s\right)\right], - v_{2,(n/m)+1} a\sin\left(\frac{3}{2}\nu t - \theta_s\right) \\ \quad - v_{2,(n/m)-1} a\sin\left(\frac{1}{2}\nu t + \theta_s\right), \end{cases} \tag{29}$$

where the coefficients $u_{i,j}$, $v_{i,j}$, the amplitude a and f are given explicitly in Appendix 2.

The response is exactly tuned to the frequency of the excitation with a phase shifted backward by the value

$$\theta_s = \frac{1}{2}\tan^{-1}\left[\frac{EG - F(D + A_\mu a_s^2)}{EF + G(D + A_\mu a_s^2)}\right], \tag{30}$$

resulting from the vanishing of Equations (26).

Figure 2 shows plots of the analytical solution of order one-half close the center c_1 obtained by the multiple scales technique of this work (Equation (29)), the one obtained by Yagazaki et al. [15] (Equation (16)) and the numerical simulation on the reduced system (15) using a Runge–Kutta method. Comparing the three approaches, it can be seen from Figure 2 that the result of our present method is nearly identical with those of the R–K method. The reason is that the method proposed here realizes a nonlinear approximation of the periodic solutions near the equilibrium point, instead of the linear one performed in [15].

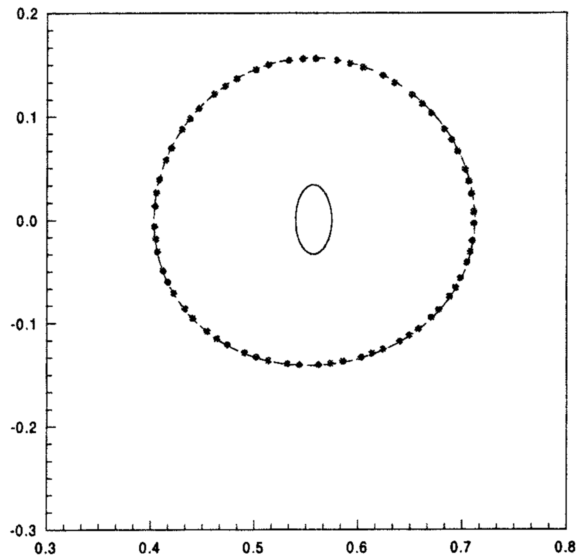

Figure 2. Comparison between numerical and analytical solutions of the reduced system (15) in the (u, v) plane for $\gamma = 1.5$, $\omega = 1.1$, $\omega_0 = 2$, $\xi = 0.5$, $\beta = 1$, $\alpha = 0.001$, $h = 0.048$ and $\nu = 2.35$. —: Approach by [15]; $\cdots$: Result of this work; - - - Numerical calculation.

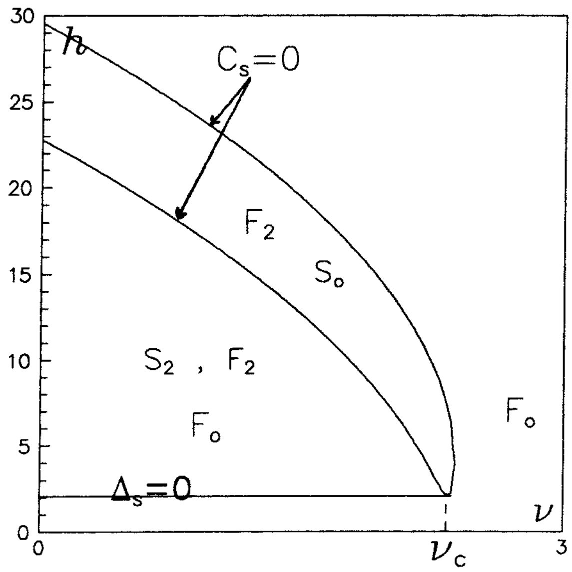

Figure 3. Bifurcation curves of subharmonic solutions of order one-half near c_1 in the (h, ν) plane for $\xi = 0.5$, $\beta = 1$, $\omega = 1.1$, $\omega_0 = 2$, $\alpha = 0.01$ and $\gamma = 1.5$.

To analyze the stability of the trivial fixed point, one investigates the solutions of the linearized equation (25). Substituting $A = (A_r + iA_i)e^{i(\tilde{\sigma}/2)T_2}$ into Equation (25), where A_r and A_i are real quantities, separating real and imaginary parts, one obtains

$$\dot{A}_r = \left(\frac{E+F}{2\Omega}\right) A_r + \left(\frac{D-G}{2\Omega}\right) A_i, \quad \dot{A}_i = -\left(\frac{D+G}{2\Omega}\right) A_r + \left(\frac{E-F}{2\Omega}\right) A_i. \tag{31}$$

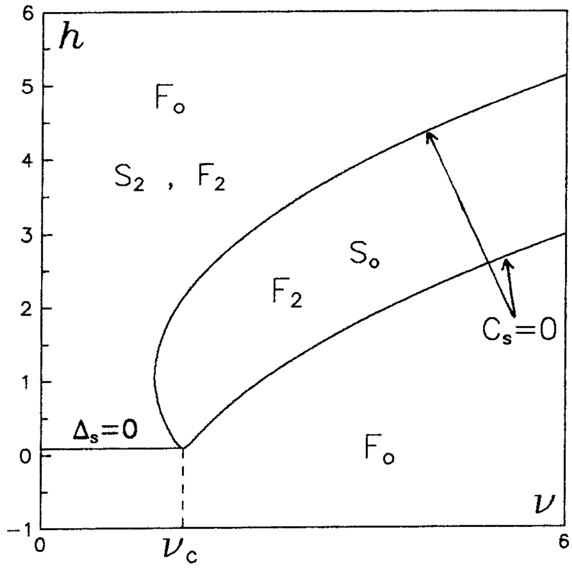

Figure 4. Bifurcation curves of subharmonic solutions of order one-half near C_3 in the (h, ν) plane for $\xi = 0.5$, $\beta = 1$, $\omega = 1.1$, $\omega_0 = 2$, $\alpha = 0.01$ and $\gamma = 1.5$.

The trivial fixed point is unstable (saddle S_0) when $C_s < 0$, otherwise it is a stable focus F_0 if Trace$(M) = E/\Omega < 0$, where M is the Jacobian of Equations (31). Representative cases are shown in Figures 3 and 4. The bifurcation curves (28) of periodic solutions of the reduced system (15) in the plane h *versus* ν for the fixed parameter values $\xi = 0.5$, $\omega = 1.1$, $\omega_0 = 2$, $\beta = 1$, $\alpha = 0.05$ and $\gamma = 1.5$ are also reported.

3. Chaotic Motions in the Reduced System

In this section we establish the criterion for homoclinic bifurcation of the reduced system (6) using Melnikov's method [17], which consists of evaluating the distance, measured along the homoclinic loop, between the stable and unstable manifolds of the hyperbolic fixed point of the associated Poincaré map. Wiggins [22] was the first to study quasi-periodically forced Duffing oscillators where Melnikov analysis could be applied. This example established the existence of chaotic motions. The procedure followed consists in reducing the study of an equation of type (1) to the study of an associated three-dimensional Poincaré map obtained by defining a three-dimensional cross-section to the four-dimensional phase space and fixing the phase of one of the regular variables. The remaining reduced three variables that start on the cross-section evolves in time under the action of the flow gererated by Equation (1) until they return to the cross-section. For detailed comments and a description of this method, see [22, 23].

We now analyze the reduced system (6) using the Melnikov technique. In this system μ is referred to as the perturbation parameter such that for $\mu = 0$, the system (6) is a completely integrable Hamiltonien system within a Hamiltonian function given by Equation (11). By using Equations (13), the Melnikov functions for $\Gamma_{\pm}$ become

$$M_\pm(t_0) = \int_{-\infty}^{+\infty} \left\{ \begin{array}{l} \left[-\frac{\gamma}{2\omega}\sqrt{2J_\pm(t)}\sin(\theta_\pm(t))\right]\left(\frac{\tilde{h}\omega}{2}\cos(\nu(t+t_0))\right) \\ -\left[\frac{\delta}{2\omega} + \frac{A_0}{4\omega}J_\pm(t) - \frac{\gamma\cos(\theta_\pm(t))}{2\omega\sqrt{2J_\pm(t)}}\right](-\tilde{\alpha}J_\pm(t)) \end{array} \right\} \mathrm{d}t. \tag{32}$$

Substituting Equations (13) into Equation (32) and evaluating the integral by the method of residues, we have (in the case $A_0 < 0$)

$$M_+(t_0) = \frac{4\pi\tilde{h}\omega\nu_0 \sinh\left(\frac{4\nu_0(\pi-\phi_0)}{|A_0|\rho}\right)}{|A_0|\sinh\left(\frac{4\nu_0\pi}{|A_0|\rho}\right)}\sin(\nu t_0) + \frac{\tilde{\alpha}}{|A_0|}(3A_0\rho - 8\delta(\pi-\phi_0)), \tag{33a}$$

$$M_-(t_0) = -\frac{4\pi\tilde{h}\omega\nu_0 \sinh\left(\frac{4\nu_0\phi_0}{|A_0|\rho}\right)}{|A_0|\sinh\left(\frac{4\nu_0\pi}{|A_0|\rho}\right)}\sin(\nu t_0) + \frac{\tilde{\alpha}}{|A_0|}(3A_0\rho + 8\delta\phi_0), \tag{33b}$$

where

$$\phi_0 = \arccos\left[\frac{c_+ + c_-}{c_+ - c_-}\right], \quad \nu_0 = 2\omega\nu \quad \text{and} \quad \rho = \sqrt{-c_+c_-}.$$

Let us define

$$R_+(A_0,\nu_0,\delta,\gamma) = \frac{|3A_0\rho - 8\delta(\pi-\phi_0)|}{4\pi\nu_0}\,\frac{\sinh\left(\frac{4\nu_0\pi}{|A_0|\rho}\right)}{\sinh\left(\frac{4\nu_0(\pi-\phi_0)}{|A_0|\rho}\right)}, \tag{34a}$$

$$R_-(A_0,\nu_0,\delta,\gamma) = \frac{|3A_0\rho + 8\delta\phi_0|}{4\pi\nu_0}\,\frac{\sinh\left(\frac{4\nu_0\pi}{|A_0|\rho}\right)}{\sinh\left(\frac{4\nu_0\phi_0}{|A_0|\rho}\right)}. \tag{34b}$$

It follows from the Melnikov theory that if

$$\frac{h\omega}{\alpha} > R_+(A_0,\nu_0,\delta,\gamma), \quad \text{or} \quad \frac{h\omega}{\alpha} > R_-(A_0,\nu_0,\delta,\gamma), \tag{35}$$

then the reduced system (6) has transverse homoclinic orbits resulting in possible chaotic dynamics. In Figure 5, we show the homoclinic bifurcation curves $h\omega/\alpha = R_\pm(A_0,\nu_0,\delta,\gamma)$ for $\xi = 2$, $\beta = 1$, $\omega = 2$, $\omega_0 = 1$ and $\gamma = 1.15$ (dashed lines). Comparison of these curves with numerical simulation (dotted line) is also shown. Figures 6–9 illustrate the combined effect of the nonlinearities and the effect of the external excitation on the extension of the possible chaotic regions in the parameter space of the oscillator (1).

4. Suppression of Chaos

In this section we devote our attention particularly to the problem of suppression of chaos in the original quasi-periodically driven oscillator (1) close to the fundamental resonance case $p = 1; q = 1$.

To analyze the suppression of chaos in the original system (1), we introduce a third harmonic parametric component into the cubic nonlinear term of the system (1) as

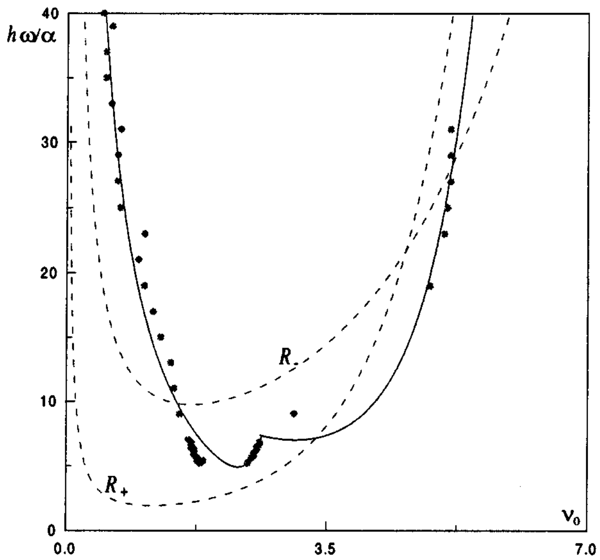

Figure 5. Homoclinic bifurcation curves (case $A_0 < 0$) in the $(h\omega/\alpha, \nu_0)$ plane for $\xi = 2$, $\beta = 1$, $\gamma = 1.15$, $\omega = 2$ and $\omega_0 = 1$.

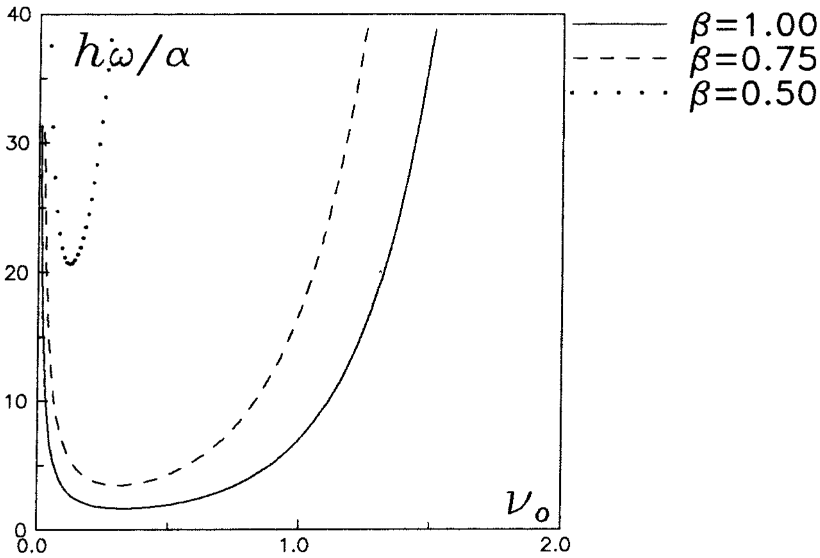

Figure 6. Quadratic nonlinearity effect on the analytical chaotic regions for $\xi = 2$, $\gamma = 0.1$, $\omega = 0.37$ and $\omega_0 = 1$.

$$\ddot{x} + \alpha\dot{x} + \omega_0^2(1 + h\cos(\nu t))x + \beta x^2 + \xi(1 + \eta\cos(\Omega t))x^3 = \gamma\cos(\omega t). \tag{36}$$

Note that the introduction of the quadratic resonant parametric perturbation for controlling chaos can also be analyzed similarly.

Let $\eta = \mu\tilde{\eta}$ and $\Omega = \varepsilon\tilde{\Omega}$, such that h and $\xi\eta$ are evaluating at the same order in both smallness parameters ε and μ. Using the same averaging process as above, we obtain the following reduced system

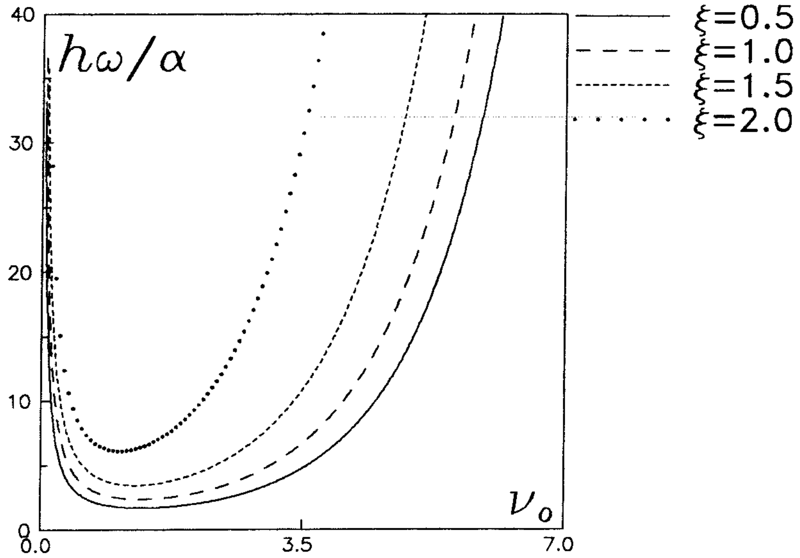

Figure 7. Cubic nonlinearity effect on the analytical chaotic regions for $\beta = 1$, $\gamma = 1.5$, $\omega = 0.7$ and $\omega_0 = 2$.

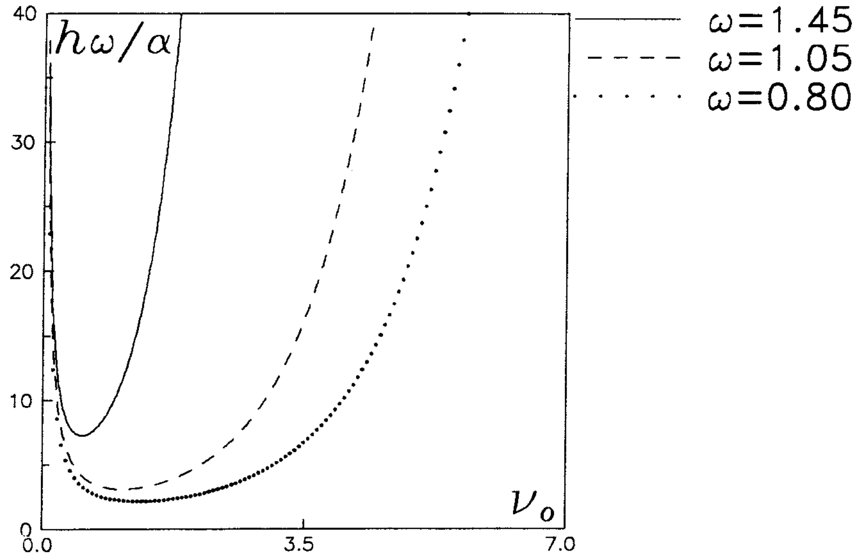

Figure 8. The effect of frequency of external excitation on the analytical chaotic regions for $\beta = 1$, $\gamma = 1.5$, $\xi = 0.5$ and $\omega_0 = 2$.

$$\left\{ \begin{array}{l} \frac{\mathrm{d}J}{\mathrm{d}t} = \frac{-\gamma}{2\omega} \sqrt{2J} \sin(\theta) - \mu(\tilde{\alpha} J), \\ \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\delta}{2\omega} + \frac{A_0}{4\omega} J - \frac{\gamma}{2\omega\sqrt{2J}} \cos(\theta) + \mu \left(\frac{\tilde{h}\omega}{2} \cos(\nu t) + \frac{3\tilde{\eta}\xi}{4\omega} J \cos(\Omega t) \right). \end{array} \right. \tag{37}$$

It can immediately be seen that by setting $\eta = 0$ in the perturbed system (37), we recover the unperturbed system (6).

Therefore, the corresponding Melnikov distance of Equation (37) for Γ_+ (in the case $A_0 > 0$) is now given by

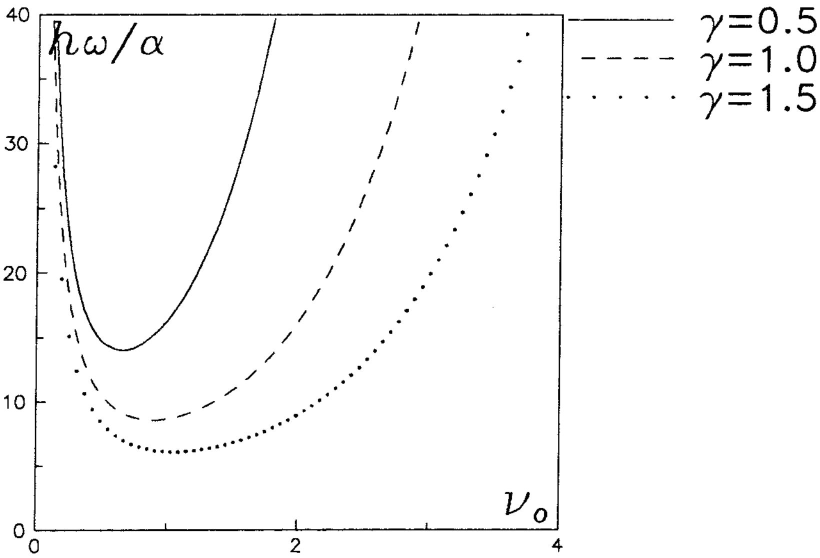

Figure 9. The effect of amplitude of external excitation on the analytical chaotic regions for $\beta = 1$, $\xi = 2$, $\omega = 0.7$ and $\omega_0 = 2$.

$$\Delta(t_0) = \Delta_0(t_0) + \frac{3\xi\eta}{4\omega}\int_{-\infty}^{+\infty}\left[-\frac{\gamma}{2\omega}\sqrt{2J_\pm(t)}\sin(\theta_\pm(t))\right](J_\pm(t)\cos(\Omega(t+t_0)))\,\mathrm{d}t. \tag{38}$$

where

$$\Delta_0(t_0) = \mu M_+(t_0) = -\frac{4\pi h\omega\nu_0\sinh\left(\frac{4\nu_0\phi_0}{|A_0|\rho}\right)}{|A_0|\sinh\left(\frac{4\nu_0\pi}{|A_0|\rho}\right)}\sin(\nu t_0) + \frac{\alpha}{|A_0|}(3A_0\rho + 8\delta\phi_0), \tag{39}$$

is the Melnikov function corresponding to the unperturbed system (6). We denote by $M^\eta_\pm(t_0)$ the integral in the right-hand side of Equation (38) related to the resonant parametric perturbation introduced in the original system (36). This integral can be performed by the method of residues for Γ_+ to yield (case $A_0 > 0$)

$$M^\eta_+(t_0) = \left\{\begin{array}{l} -\frac{6\pi\xi\eta\Omega_0 j_2\sinh\left(\frac{4\Omega_0\phi_0}{|A_0|\rho}\right)}{|A_0|\omega\sinh\left(\frac{4\Omega_0\pi}{|A_0|\rho}\right)} \\ +\frac{3\pi\xi\eta\Omega_0}{2|A_0|\omega}\left[\frac{8\Omega_0\cosh\left(\frac{4\Omega_0\phi_0}{|A_0|\rho}\right)}{|A_0|\sinh\left(\frac{4\Omega_0\pi}{|A_0|\rho}\right)} - (c_+ + c_-)\frac{\sinh\left(\frac{4\Omega_0\phi_0}{|A_0|\rho}\right)}{\sinh\left(\frac{4\Omega_0\pi}{|A_0|\rho}\right)}\right] \end{array}\right\}\sin(\Omega t_0), \tag{40}$$

where $\Omega_0 = 2\omega\Omega$. With obvious notation, we can rewrite Equation (38) in the following form

$$\Delta(t_0) = A(\nu_0)\sin(\nu t_0) + B(\Omega_0)\sin(\Omega t_0) + C, \tag{41}$$

where the quantities $A(\nu_0)$, $B(\Omega_0)$, and C are given according to Equations (39) and (40).

To afford a better understanding of the discussion below, in Figure 10, we have plotted the functions $A(\nu_0)$, $B(\Omega_0)$ and C. First, assume that the system (1) is in a chaotic state for which homoclinic transverse orbits exist. Hence, the Melnikov distance Δ_0, corresponding to the

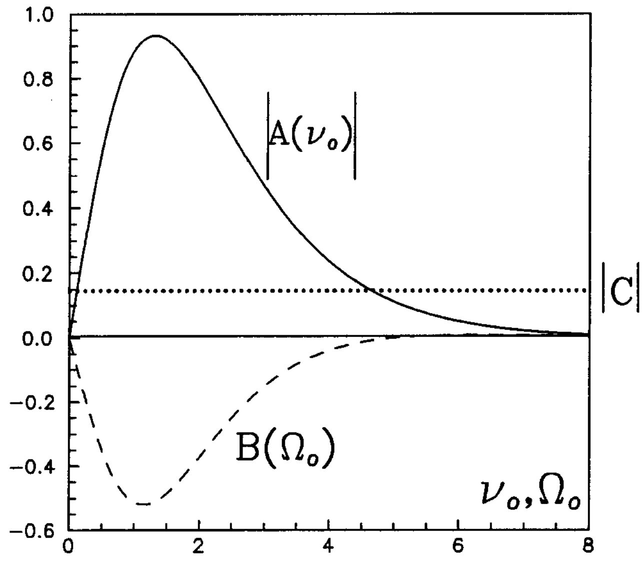

Figure 10. Plot of functions $A(\nu_0)$ and $B(\Omega_0)$ for $\omega = 3$, $\omega_0 = 2$, $\gamma = 1.5$, $\beta = 1$, $\xi = 0.5$, $\eta = 0.1$, $h = 0.088$ and $\alpha = 0.005$.

unperturbed system (6), changes sign at some t_0. In this case, the condition to be satisfied is $|A(\nu_0)| - |C| = d > 0$. Moreover, if $|B(\Omega_0)| < d$, it is easy to see from Equation (41) that for some t_0, $\Delta(t_0)$ will change sign and then the situation remains unchanged. Furthermore, if the frequency of the resonant parametric perturbation Ω is in resonance with the driven frequency ν, a necessary and sufficient condition [7] for $\Delta(t_0)$ to be positive for all t_0 is given by

$$|B(\Omega_0)| > d. \tag{42}$$

More precisely, this condition provides a domain on the parameter plane (η, Ω) of the resonant parametric perturbation on which the chaotic behavior in the original system is suppressed.

In Figure 11, and according to the condition (42), we illustrate the regions, in the parameter plane (η, Ω_0), in which the chaotic behavior of the system (1) can be suppressed. The region I corresponds to a regular motion for which condition (42) is satisfied. In the region II the system still exhibits chaotic behavior.

5. Numerical Simulations

To support our analytical prediction of the control of chaos in Equation (1), we have performed computer simulations of the reduced system (37) which can be written in the Cartesian form

$$\begin{cases} \dot{u} = \frac{\delta}{2\omega}\nu - \frac{\alpha}{2}u + \left[\frac{A_0}{8\omega} + \frac{3\eta\xi}{8\omega}\cos(\Omega t)\right](u^2+\nu^2)\nu + \frac{h\omega}{2}\cos(\nu t)\nu, \\ \dot{\nu} = \frac{\gamma}{2\omega} - \frac{\delta}{2\omega}u - \frac{\alpha}{2}\nu - \left[\frac{A_0}{8\omega} + \frac{3\eta\xi}{8\omega}\cos(\Omega t)\right](u^2+\nu^2)u - \frac{h\omega}{2}\cos(\nu t)u. \end{cases} \tag{43}$$

These differential equations are integrated numerically with those of the dynamics in tangent space of the system (43) by using the fourth-order Runge–Kutta method. We then extract

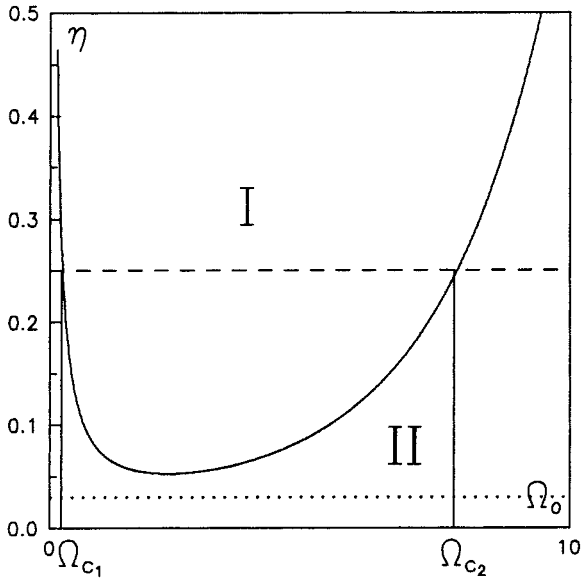

Figure 11. Suppression of chaos threshold in the (η, Ω_0) plane for $\omega = 2$, $\gamma = 1.15$, $\omega_0 = 1$, $\beta = 1$, $\alpha = 0.1$, $\nu = 0.5$, $h = 0.574$ and $\xi = 2$.

the Lyapunov characteristic exponents by applying the Gram–Schmidt orthonormalization procedure [24]. The units for the exponents are bits/sec.

First, we choose some arbitrary set of parameters for the unperturbed part ($\eta = 0$) of Equations (43), such that the Melnikov criterion is largely satisfied. We ensure numerically that chaos is actually present after having eliminated a long transient. This character is also confirmed by the positiveness of the leading Lyapunov exponent. For the set of parameters $\eta = 0$, $\xi = 2$, $\beta = 1$, $\alpha = 0.1$, $\omega_0 = 1$, $\omega = 2$, $\gamma = 1.15$, $h = 0.574$ and $\nu = 0.5$, the solution of Equations (43) is chaotic with a leading Lyapunov exponent equal to about $\lambda_1 = 0.057$.

Consider now the effect produced by adding a small perturbation with $\eta = 0.25$ and $\eta = 0.03$, respectively. The relevant maximal Lyapunov exponent λ *versus* Ω is presented in Figures 12 and 13. The important conclusion which follows from such dependences is twofold. First, the leading Lyapunov exponent vanishes and becomes negative around some multiples and submultiples of the forcing frequency ν, and thus the regular motion is actually recovered. Second, the values of $\lambda(\Omega)$, computed off-resonance with $\eta \neq 0$, are close to the unperturbed ($\eta = 0$) value of λ. The dotted line corresponds to the unperturbed value of λ obtained at $\eta = 0$.

The numerical computations show that the analytical prediction given by the condition (42) overestimates the threshold for the value of η. Figure 11 reports that, for a given set of parameters of the unperturbed system, the value $\eta = 0.03$ (dotted line) is not sufficient to suppress chaos while numerical simulations (Figure 13) already give $\lambda(\Omega) = 0$ for the frequencies $\Omega = (3/5)\nu$, $(6/5)\nu$ and $(13/5)\nu$. For $\eta = 0.25$ (dashed line) the expected resonant frequencies to eliminate chaos belong to the range $\Omega_{c1} \leq \Omega \leq \Omega_{c2}$ (see Figure 11; $\Omega_{c1} = 0.05$ and $\Omega_{c2} = 1.94$). Numerically, one observes in Figure 12 several values of Ω, belonging to the interval $[\Omega_{c1}; \Omega_{c2}]$, for which the leading Lyapunov exponent λ is negative; for instance $\omega = (1/2)\nu$, $(3/5)\nu$, ν, $(49/25)\nu$, $(5/2)\nu$ and 3ν. These numerical results confirm our analytical predictions.

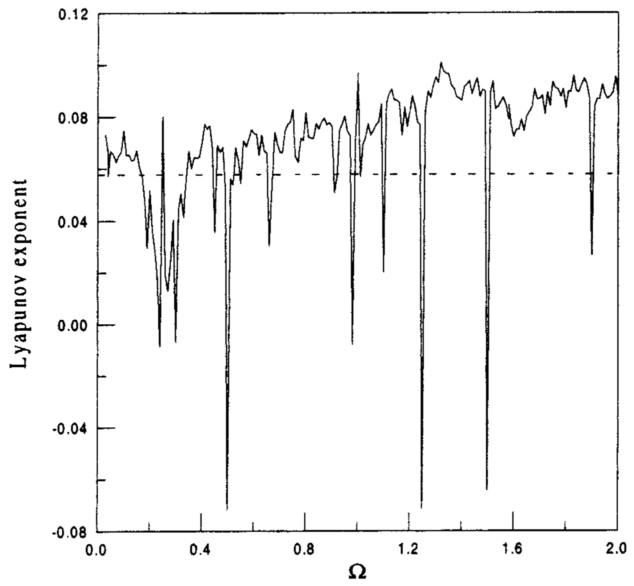

Figure 12. The leading Lyapunov characteristic exponent λ is reported *versus* Ω. This result corresponds to $\omega = 2$, $\gamma = 1.15$, $\beta = 1$, $\xi = 2$, $\omega_0 = 1$, $h = 0.574$, $\nu = 0.5$ and $\eta = 0.25$. The dashed line refers to $\eta = 0$ (unperturbed case) for which $\lambda = 0.057$.

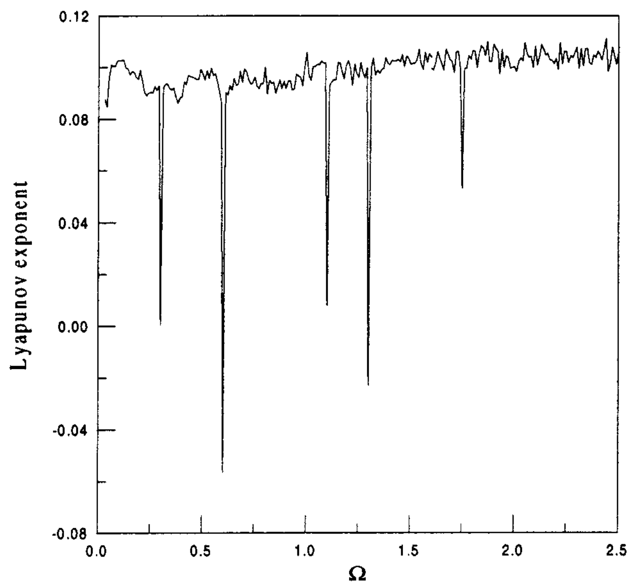

Figure 13. The leading Lyapunov characteristic exponent λ is reported *versus* Ω. This result corresponds to $\omega = 2$, $\gamma = 1.15$, $\beta = 1$, $\xi = 2$, $\omega_0 = 1$, $h = 0.574$, $\nu = 0.5$ and $\eta = 0.03$.

6. Conclusions

The paper is concerned with the dynamic analysis of controlling chaos in a one-degree-of-freedom system with quadratic and cubic nonlinearities subjected to external and parametric excitations having incommensurate frequencies.

To construct an analytical approximation of the quasi-periodic solutions, a new method requiring two steps is proposed. The first step mainly consists of transforming the original quasi-periodic system to a reduced periodically driven one using the generalized averaging method. This method can be viewed as an adaptation to quasi-periodic systems of the technique developed by Bogolioubov and Mitropolsky for periodically driven ones. Our original contribution is the application of the multiple scales method in a second step to the reduced system in order to construct an asymptotic expansion of the quasi-periodic solutions. The advantage of this method is the realization of a nonlinear approximation of periodic solutions of the reduced periodically driven system near the equilibrium points instead of the linear one.

Furthermore, the perturbation methods are usually applied to study the solutions close to one resonance. In contrast, the proposed method provides the construction of an asymptotic expansion to investigate quasi-periodic solutions close to several resonance points simultaneously. For instance, the formulation allows one to investigate the two resonance cases $p = 1; q = 1$ and $p = 1; q = 2$. In addition, near the fundamental resonance $p = 1; q = 1$ considered here, three possible resonance cases $n = 1; m = 1$, $n = 2; m = 1$ and $n = 3; m = 1$ can also be investigated simultaneously.

The second contribution of this work is the realization of a suitable system control of chaos by introducing a third harmonic parametric component into the cubic term. First, the Melnikov technique is applied to the reduced averaged periodically driven system to predict the occurrence of chaotic responses. By introducing a nonlinear parametric perturbation, we have shown that the control of the chaotic dynamic in quasi-periodically excited oscillators can be acheived. It has been shown that this complex dynamic can become regular provided that the amplitude parameter of the resonant parametric perturbation is larger than some critical value, and the frequency of the perturbation is in resonance with the frequency of the forcing of the reduced system. Numerical support confirms the analytical predictions.

Appendix 1

$$C_1^A = -\Omega f_{46}, \quad C_{(n/m)-1}^{\bar{A}e^{i\tilde{\sigma}T_2}} = f_{48},$$

$$\begin{aligned} Q_1^A = {} & f_5 f_6 - \Omega^2 f_1 f_2 - \Omega^2 f_3 f_4 - \frac{\Omega^2 A_0}{8k\omega} f_{9R} + f_{14R} + \Omega^2 f_{16} f_{17} - \Omega^2 f_{18} f_{19} \\ & + f_{20} f_{21} + f_{20} f_{22} - \frac{(\Omega A_0)^2 \tilde{\alpha} u_0}{(8k\omega)^2} f_{23} - f_{26} f_{49} - \frac{\tilde{\alpha}}{2} f_{49} - \frac{1}{4\Omega^2} f_{48}^2 + \frac{1}{2} f_{46} f_{49}, \end{aligned}$$

$$Q_1^{\bar{A}A^2} = \frac{\Omega^2 A_0}{8k\omega} (f_{11} - f_{9V}) + f_{14V} + f_{15},$$

$$Q_1^{\bar{A}e^{i\tilde{\sigma}T_2}} = \frac{1}{2\Omega} f_{26} f_{48} + \frac{\tilde{\alpha}}{4\Omega} f_{48} - \frac{1}{4\Omega} f_{46} f_{48} + \frac{1}{2\Omega} f_{48} f_{49},$$

$$Q_{(n/m)-1}^A = \frac{4k\omega}{A_0} f_{29} f_{48} - \frac{A_0 u_0}{8\omega} f_{18} f_{48} - \left(\frac{n}{m} - 1\right) \frac{A_0}{8k} \left\{ \frac{\tilde{h}}{8} + \frac{A_0 u_0}{8\omega^2} f_{20} \right\} f_{48},$$

$$Q_{(n/m)-1}^{\bar{A}e^{i\tilde{\sigma}T_2}} = \frac{4k\omega\Omega}{A_0} f_{29} f_{46} + \Omega f_4 f_{31} - \Omega f_{18} f_{33} + \frac{\Omega A_0 u_0}{4\omega} f_{18} f_{49} + \frac{\Omega A_0}{8k\omega} f_{41}$$

$$+\,\Omega(f_{42}+f_{44})+\frac{A_0^2u_0\Omega}{64k\omega^2}\left\{2\left(\frac{n}{m}-1\right)f_{49}+\tilde{\alpha}\right\}f_{20},$$

$$E=\mu C_1^A,\quad F=\mu^2(Q_1^{\bar{A}e^{i\tilde{\sigma}T_2}}+Q_{(n/m)-1}^{\bar{A}e^{i\tilde{\sigma}T_2}}),\quad G=\mu C_1^{\bar{A}e^{i\tilde{\sigma}T_2}},$$

$$H=\mu^2(Q_1^A+Q_{(n/m)-1}^A),\quad A_\mu=\mu^2\frac{Q_1^{\bar{A}A^2}}{4},\quad \tilde{H}_0=\frac{(1-\delta_{n,m})}{\left(1-\frac{n^2}{m^2}\right)}\frac{\tilde{f}}{2\Omega^2}.$$

The quantities k, $\tilde{f}$ and f_j are given by

$$k=\frac{A_0}{8\omega}\left(\frac{\delta}{2\omega}+\frac{A_0}{8\omega}u_0^2\right),\quad \tilde{f}=\frac{\tilde{h}\omega u_0}{2}\left(\frac{\delta}{2\omega}+\frac{A_0}{8\omega}u_0^2\right),$$

$$f_1=\frac{\tilde{h}\omega}{4}-\left(\frac{n}{m}+1\right)\frac{A_0u_0}{4\omega}\tilde{H}_0,$$

$$f_2=\frac{A_0}{8k\omega}\left\{\left(\frac{\frac{n}{m}+1}{\Omega^2\left[1-\left(\frac{n}{m}+1\right)^2\right]}\left[\frac{\gamma}{\omega}\left(3-\frac{n}{m}\left(\frac{A_0\Omega}{8k\omega}\right)^2\right)+\left(\frac{n}{m}+1\right)^2\frac{A_0u_0\Omega^2}{4k\omega}\right]\right.\right.$$

$$\left.+\left(\frac{n}{m}+1\right)\frac{A_0u_0}{4k\omega}\right)\frac{A_0}{8\omega}\tilde{H}_0$$

$$\left.-\frac{\frac{n}{m}+1}{\Omega^2\left[1-\left(\frac{n}{m}+1\right)^2\right]}\left[\frac{\tilde{f}}{2u_0}+\left(\frac{n}{m}+1\right)\frac{A_0\Omega^2\tilde{h}}{32k}\right]-\frac{A_0\tilde{h}}{32k}\right\},$$

$$f_3=\frac{\tilde{h}\omega}{4}+\left(\frac{n}{m}-1\right)\frac{A_0u_0}{4\omega}\tilde{H}_0,\quad f_4=-\left(\frac{A_0}{8k\omega}\right)^2\left[\left(\frac{n}{m}-1\right)\frac{A_0u_0}{4\omega}\tilde{H}_0+\frac{\tilde{h}\omega}{4}\right],$$

$$f_5=u_0\left[6k-\frac{n}{2km}\left(\frac{A_0\Omega}{4\omega}\right)^2\right]\tilde{H}_0-\frac{\tilde{f}}{2u_0},$$

$$f_6=\frac{1}{\Omega^2\left[1-\left(\frac{n}{m}+1\right)^2\right]}\left\{\left[\frac{\gamma}{\omega}\left(3-\frac{n}{m}\left(\frac{A_0\Omega}{8k\omega}\right)^2\right)+\left(\frac{n}{m}+1\right)^2\frac{A_0u_0\Omega^2}{4k\omega}\right]\frac{A_0}{8\omega}\tilde{H}_0\right.$$

$$\left.-\left[\frac{\tilde{f}}{2u_0}+\left(\frac{n}{m}+1\right)\frac{A_0\Omega^2\tilde{h}}{32k}\right]\right\},$$

$$f_7=u_0\left[6k+\frac{n}{2km}\left(\frac{A_0\Omega}{4\omega}\right)^2\right]\tilde{H}_0-\frac{\tilde{f}}{2u_0},$$

$$f_8=(\tilde{\alpha}u_0)^2A_0\left\{\left(\frac{A_0}{8k\omega}\right)^2\left[\frac{1}{32\omega}-\frac{A_0u_0}{(16\Omega)^2\omega^3}\right]-\frac{1}{16\omega\Omega^2}\right\},$$

$$f_9=\frac{H_0A_0}{4\omega}\left\{\tilde{H}_0\left[\left(1-\frac{A_0u_0\gamma}{8(\omega\Omega)^2}\right)\left(1+\left(\frac{n}{m}\frac{A_0\Omega}{8k\omega}\right)^2\right)-\frac{A_0u_0\gamma}{(2\omega\Omega)^2}\right]+\frac{\tilde{f}}{\Omega^2}\right\},$$

$$f_{9R}=f_8+f_9,\quad f_{9V}=\frac{A_0}{4\omega}\left\{\left(1-\frac{A_0u_0\gamma}{8(\omega\Omega)^2}\right)\left[1+\left(\frac{A_0\Omega}{8k\omega}\right)^2\right]-\frac{A_0u_0\gamma}{(2\omega\Omega)^2}\right\},$$

$$f_{11}=\frac{A_0u_0}{4\omega}\left\{\frac{1}{k}\left(\frac{A_0u_0}{8\omega}\right)^2-\frac{1}{3\Omega^2}\left[\frac{A_0\gamma}{2\omega^2}\,\frac{\delta}{4\delta+A_0u_0^2}+4ku_0\right]\right\},$$

$$f_{12}=\frac{\tilde{\alpha}^2u_0^2}{2}\left\{-\frac{A_0}{16\omega}+\frac{3k}{\Omega^2}\left[1+\frac{A_0\gamma u_0}{16\omega^2}\left(\frac{A_0}{8k\omega}\right)2\right]\right\},$$

$$f_{13}=k\tilde{H}_0\left\{\tilde{H}_0\left[\left(\frac{3A_0u_0\gamma}{(2\omega\Omega)^2}-2\right)\left(1+\left(\frac{n}{m}\,\frac{A_0\Omega}{8k\omega}\right)^2\right)+\frac{3A_0u_0\gamma}{2(\omega\Omega)^2}\right]-\frac{6\tilde{f}}{\Omega^2}\right\},$$

$$f_{14R}=f_{12}+f_{13},\quad f_{14V}=k\left\{\left(\frac{3A_0u_0\gamma}{(2\omega\Omega)^2}-2\right)\left[3+\left(\frac{A_0\Omega}{8k\omega}\right)^2\right]+4\right\},$$

$$f_{15}=u_0\left[3+2\left(\frac{A_0\Omega}{8k\omega}\right)^2\right]$$
$$\times\left\{\frac{u_0}{2}\left(\frac{A_0}{4\omega}\right)^2-\frac{2k}{3\Omega^2}\left[\frac{A_0\gamma}{2\omega^2}\,\frac{\delta}{4\delta+A_0u_0^2}+4ku_0\right]\right\}-\left(\frac{A_0u_0}{4\omega}\right)^2,$$

$$f_{16}=-\frac{n}{m}\,\frac{A_0\tilde{H}_0}{8k\omega},\quad f_{17}=\frac{A_0}{4\omega}\left[1+\frac{n}{m}\left(\frac{A_0\Omega}{8k\omega}\right)^2\right]\tilde{H}_0,$$

$$f_{18}=\frac{n}{m}\,\frac{A_0\tilde{H}_0}{8k\omega},\quad f_{19}=\frac{A_0}{4\omega}\left[\frac{n}{m}\left(\frac{A_0\Omega}{8k\omega}\right)^2-1\right]\tilde{H}_0,$$

$$f_{20}=-\tilde{H}_0,\quad f_{21}=2k\left[1+\frac{n}{m}\left(\frac{A_0\Omega}{8k\omega}\right)^2\right]\tilde{H}_0,\quad f_{22}=2k\left[1-\frac{n}{m}\left(\frac{A_0\Omega}{8k\omega}\right)^2\right]\tilde{H}_0,$$

$$f_{26}=\tilde{\alpha}k\left(\frac{A_0u_0}{8k\omega}\right)^2,\quad f_{29}=\frac{A_0}{32k\omega}\left[\tilde{h}\omega+\frac{A_0u_0}{\omega}\left(\frac{n}{m}-1\right)\tilde{H}_0\right],$$

$$f_{31}=\tilde{\alpha}k\left(\frac{A_0u_0}{8k\omega}\right)^2,\quad f_{33}=\left(\frac{n}{m}-1\right)\frac{A_0u_0\tilde{\alpha}}{8\omega}\left(\frac{A_0\Omega}{8k\omega}\right)^2,$$

$$f_{36}=\frac{n}{m}\,\frac{2\tilde{\alpha}}{\tilde{f}}\,\tilde{H}_0\left\{\tilde{H}_0\left[1+\frac{A_0u_0\gamma}{(4\omega)^2}\left(\frac{A_0}{8k\omega}\right)^2\right]+\frac{A_0u_0}{64k}\left(\tilde{h}-\frac{A_0u_0}{\omega^2}\,\tilde{H}_0\right)\right\},$$

$$f_{39}=\frac{\tilde{\alpha}}{32k}\left[\frac{A_0u_0}{2}\left(\frac{A_0u_0}{\omega^2}\,\tilde{H}_0-\tilde{h}\right)-\tilde{H}_0\right]-\frac{n}{m}\,\Omega^2 f_{36},$$

$$f_{41}=\left(\frac{n}{m}-1\right)\frac{A_0u_0\Omega^2}{4\omega}\left[f_{36}+\frac{n}{m}\,\frac{\tilde{\alpha}}{2}\left(\frac{A_0}{8k\omega}\right)^2\tilde{H}_0\right]+2u_0kf_{39},$$

$$f_{42} = \frac{A_0 u_0}{4\omega}\left[\left(\frac{n}{m}-1\right) f_{39} - \frac{n}{m}\,\frac{A_0\tilde{\alpha}}{16k\omega}\,\tilde{H}_0\right] - 6u_0 k f_{36},$$

$$f_{44} = \left(\frac{n}{m}-1\right)\frac{A_0}{8k\omega}\left\{\frac{\tilde{h}\omega}{4}\, f_{49} - \frac{A_0 u_0\tilde{\alpha}}{8\omega}\left[1+\frac{n}{m}\left(\frac{A_0\Omega}{8k\omega}\right)^2\right]\tilde{H}_0\right\} - \frac{4k\omega\tilde{\alpha}}{A_0}\, f_{29},$$

$$f_{46} = \tilde{\alpha}\left\{1+\frac{u_0}{k}\left(\frac{A_0}{8\omega}\right)^2\left[\frac{A_0\gamma}{16k\omega^2}-u_0\right]\right\},$$

$$f_{48} = \frac{A_0\gamma}{8\omega^2}\left[3+\frac{n}{m}\left(\frac{A_0\Omega}{8k\omega}\right)^2\right]\tilde{H}_0 + \left(\frac{n}{m}-1\right)\frac{A_0\Omega^2}{4\omega}\left[\frac{\tilde{h}\omega}{4}+\left(\frac{n}{m}-1\right)u_0\tilde{H}_0\right] - \frac{\tilde{f}}{2u_0},$$

$$f_{49} = \frac{\tilde{\alpha}}{2}\left[1-\frac{1}{2k}\left(\frac{A_0 u_0}{4\omega}\right)^2\right] - \frac{1}{2}\, f_{46}.$$

Appendix 2

$$u_{2,0} = -\left(\frac{\alpha}{2}\right)^2 u_0 - \frac{A_0\gamma}{8\omega^2}\left\{\left[3+\left(\frac{A_0\Omega}{4k\omega}\right)^2\right]H_0^2 + \left[3+\left(\frac{A_0\Omega}{8k\omega}\right)^2\right]\frac{a^2}{4}\right\} + \frac{f}{u_0}\,H_0,$$

$$u_{2,2} = \frac{A_0\gamma}{16\omega^2}\left[\left(\frac{A_0\Omega}{8k\omega}\right)^2-3\right] - \frac{u_0}{k}\left(\frac{A_0\Omega}{4\omega}\right)^2, \quad u_{2,(n/m)} = 2\Omega\alpha\left[H_0+\frac{A_0 u_0}{64k}\,h\right],$$

$$u_{2,(2n/m)} = \frac{A_0\gamma}{16\omega^2}\left[\left(\frac{A_0\Omega}{4k\omega}\right)^2-3\right]H_0^2 - \frac{f}{2u_0}\,H_0 + \frac{A_0\Omega^2}{4k\omega}\left(h\omega - \frac{A_0 u_0}{\omega}\,H_0\right)H_0,$$

$$u_{2,(n/m)+1} = \frac{A_0\gamma}{8\omega^2}\left[3-2\left(\frac{A_0\Omega}{8k\omega}\right)^2\right]H_0 - \frac{f}{2u_0} + \frac{3A_0\Omega^2}{32k\omega}\left(\frac{3A_0 u_0}{\omega}\,H_0 - h\omega\right),$$

$$v_{2,1} = \frac{A_0\alpha}{16k\omega}\left\{\left[1-\frac{1}{2k}\left(\frac{A_0 u_0}{4\omega}\right)^2\right] - \left\{1+\frac{u_0}{k}\left(\frac{A_0}{8\omega}\right)^2\left[\frac{A_0\gamma}{16k\omega^2}-u_0\right]\right\}\right\},$$

$$v_{2,1b} = \frac{A_0}{16k\omega\Omega}\left\{\frac{A_0\gamma}{8\omega^2}\left[3+2\left(\frac{A_0\Omega}{8k\omega}\right)^2\right]H_0 + \frac{A_0\Omega^2}{4\omega}\left[\frac{\tilde{h}\omega}{4}+u_0 H_0\right] - \frac{f}{2u_0}\right\},$$

$$v_{2,2} = \frac{A_0}{8k\omega}\left\{\frac{2}{3\Omega}\,u_{2,2} + \frac{\Omega u_0 A_0^2}{32k\omega^2}\right\},$$

$$v_{2,(n/m)} = \frac{A_0}{8k\omega}\left\{\frac{2}{3\Omega}u_{2,(n/m)} + \alpha\left\{\left[\frac{1}{k}\left(\frac{A_0 u_0}{8\omega}\right)^2-\frac{1}{2}\right]H_0 - \frac{A_0 u_0}{64k}\,h\right\}\right\},$$

$$v_{2,(2n/m)} = \frac{A_0}{8k\omega}\left\{-\frac{4}{15\Omega}\,u_{2,(2n/m)} + \frac{A_0\Omega}{16k}\left[h-\frac{A_0 u_0}{\omega^2}\,H_0\right]H_0\right\},$$

$$\nu_{2,(n/m)+1} = \frac{A_0}{8k\omega}\left\{-\frac{3}{8\Omega}u_{2,(n/m)+1} + \frac{A_0\Omega}{32k}\left[\frac{3A_0u_0}{\omega^2}H_0 - h\right]\right\},$$

$$\nu_{2,(n/m)-1} = \frac{A_0}{8k\omega}\left\{\frac{A_0\Omega}{32k}\left[\frac{A_0u_0}{\omega^2}H_0 + h\right]\right\},$$

where $a = \mu a_s$, $H_0 = \mu\tilde{H}_0$ and $f = \mu\tilde{f}$.

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