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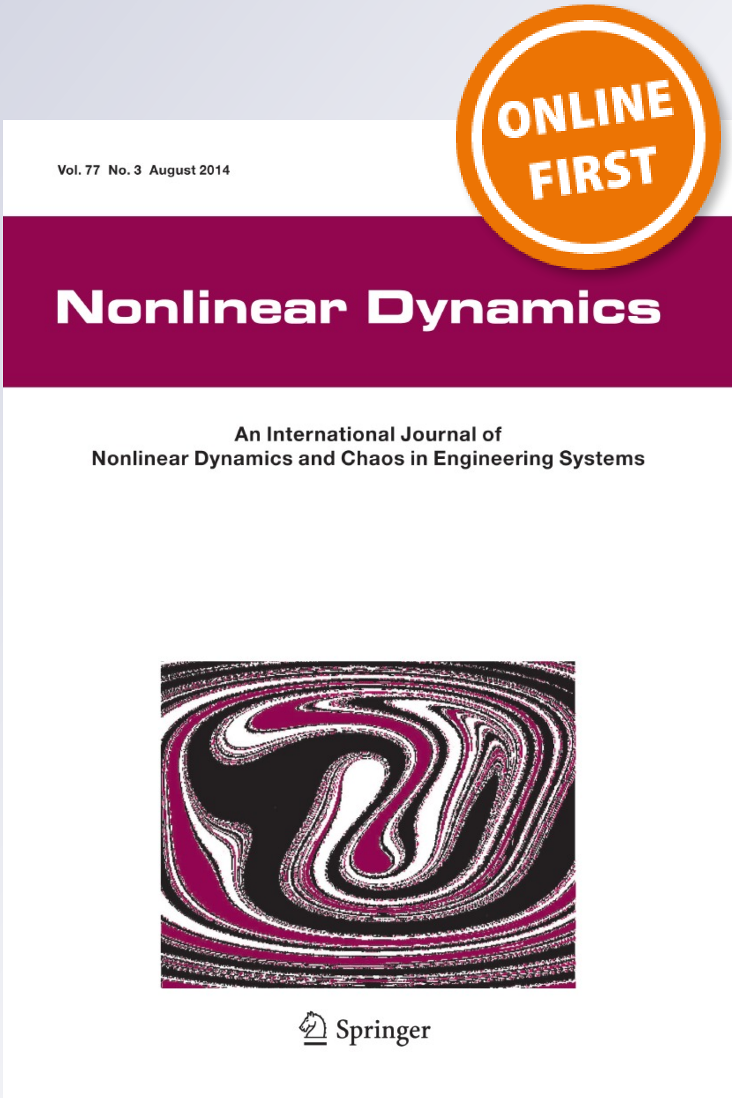
Nonlinear Dynamics
An International Journal of Nonlinear Dynamics and Chaos in Engineering Systems

ISSN 0924-090X

Nonlinear Dyn
DOI 10.1007/s11071-014-1604-x



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ORIGINAL PAPER

Heteroclinic connections in the 1:4 resonance problem using nonlinear transformation method

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Received: 29 July 2013 / Accepted: 15 July 2014

Abstract In this paper, the method of nonlinear time transformation is applied to obtain analytical approximation of heteroclinic connections in the problem of stability loss of self-oscillations near 1:4 resonance. As example, we consider the case of parametric and self-excited oscillator near the 1:4 subharmonic resonance. The method uses the unperturbed heteroclinic connection in the slow flow to determine conditions under which the perturbed heteroclinic connection persists. The results show that for small values of damping, the nonlinear time transformation method can predict well both the *square* and *clover* heteroclinic connection near the 1:4 resonance. The analytical finding is confirmed by comparisons to the results obtained by numerical simulations.

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Published online: 03 August 2014

Keywords 1:4 Resonance · Heteroclinic bifurcation · Nonlinear transformation · Perturbation analysis

1 Introduction

Bifurcation analysis of heteroclinic orbits near the 1:4 resonance is considered as one of the most intriguing and unsolved problem in nonlinear dynamical systems for which a rigorous proof is still incomplete [1]. This resonance problem can be regarded as a transition between the strong 1:q resonance ($q = 1, 2, 3$) and the weak 1:q resonance ($q = 5, 6, \ldots$) for which the dynamic and the bifurcation of phase portraits are well established [1,2]. It is well known that the bifurcation problem of periodic orbits close to 1:4 resonance can be reduced to the analysis of the normal form differential equation [1]

$$\dot{z} = \varepsilon z + Az|z|^2 + Bz^3, \quad (1)$$

where z is a complex variable and ε, A, B are complex parameters. Since the pioneer work by Arnold [1] on this issue, several authors analyzed this bifurcation problem showing a rich and various bifurcation situations with complicated and various phase portraits including precisely homoclinic and heteroclinic connections of different types [1–8].

It is worth noticing that homoclinic and heteroclinic orbits are of great importance from applied points of view [9]. The existence of a homoclinic or heteroclinic orbit in a system of ordinary differential equations implies the existence of a coherent structure such as a soliton in certain partial differential equations [10–12]. For instance, they form the profiles of travelling wave solutions in reaction–diffusion problems [13]. In static-dynamics analogies, a homoclinic orbit corresponds to a spatially localized post-buckling state [14]. Additional applications can be found in [15–17].

As practical example to illustrate the significance of the heteroclinic bifurcations near strong resonances is the onset of various types of synchronization for systems of globally coupled phase oscillators that can be encountered in physics and biology [18–20]. Such systems have been found to exhibit robust "slow switching" oscillations that are caused by the presence of robust heteroclinic attractors [21].

The 1:4 resonance problem may also arise in self-excited and harmonically driven nonlinear oscillators for which their slow flow reduces precisely to the normal form (1) near this resonance. Therefore, investigating the dynamic of a slow flow near the 1:4 subharmonic resonance arising in engineering applications is equivalent to study the dynamic of the normal form (1). For instance, the example of the forced van der Pol equation near the 1:4 resonance was considered in [3], while the self-excited nonlinear Mathieu equation was treated in [4,5] near this resonance.

One of the challenging problems to tackle for this resonance is the analytical capture of the heteroclinic bifurcation, which occur when pairs of saddles form connections resulting in the stability loss of self-oscillations near the resonance. Two types of heteroclinic bifurcation of dimension one occur near the 1:4 resonance: the *square* and the *clover* connections. The *square* connection leaves the stable subharmonic cycle outside the connection, while the *clover* connection surrounds the stable subharmonic cycle. Due to difficulties encountered in obtaining rigorous analytical treatment of heteroclinic bifurcations near this resonance, only numerical simulation [4,5] including continuation methods [6–8,22] has been performed. Such difficulties come from the fact that heteroclinic connection near such a resonance cannot be reduced into the problem of homoclinic bifurcation in some plenary autonomous system by rescaling [23]. Motivated by the lack of rigorous analytical treatment to this problem, perturbation methods have been developed in a series of paper to construct approximation of such bifurcations [24–27]. The idea used to capture such bifurcations is based on the collision criterion [25] between the periodic solution (slow flow limit cycle), and the saddles involved in the bifurcation. This criterion was applied recently [28] to obtain analytical approximation to the *square* heteroclinic connection in the 1:4 resonance problem considering a parametric and self-excited nonlinear oscillator. The comparison between the analytical finding and the results of numerical simulations shows a good match indicating that the collision criterion can be exploited to capture such bifurcations. In this study [28], only the *square* connection, in which the limit cycle born by Hopf bifurcation disappears from inside the connection, has been tackled. Instead, the analytical prediction of the *clover* connection for which the limit cycle disappears from outside the connection has not been addressed. This is because analytical approximation of the slow flow limit cycle born outside the connection far from the origin cannot be obtained when using trigonometric functions. This issue is still an open challenging problem, and (probably) the use of elliptic functions may be the key to the problem.

The difficulty in capturing the *clover* connection using the collision criterion and trigonometric functions in approximating the periodic orbit motivated the present work for which an alternative method is proposed. The strategy consists in applying the so-called nonlinear time transformation method [29] based on the perturbation of the unperturbed heteroclinic connection of the slow flow [or the normal form (1)]. The approximation of bifurcation is then obtained by determining conditions under which the perturbed heteroclinic connection persists.

The paper is organized as follows. In Sect. 2, we give the condition for the slow flow to be a Hamiltonian system. This provides the unperturbed zero-order approximation of a heteroclinic connection. In Sect. 3, the perturbation of the zero-order approximation of the heteroclinic connection is performed, and the first-order approximation of the *square* and *clover* heteroclinic connections is obtained and compared to results given by numerical simulations. The last section concludes the work.

2 Zero-order heteroclinic connection

Heteroclinic bifurcation near the 1:4 resonance can occur, for instance, in the following nonlinear parametric and self-excited oscillator [28]

$$\ddot{x} + \omega_0^2(1 + h\cos\omega t)x - (\alpha - \beta x)\dot{x} - cx^2 = 0, \quad (2)$$

where ω_0 is the natural frequency, $\alpha,\ \beta$ are the damping coefficients, c is the quadratic nonlinear component, and $h,\ \omega$ are, respectively, the amplitude and the frequency of the parametric excitation. Assuming the 1:4 resonance condition $\omega_0^2 = \left(\frac{\omega}{4}\right)^2 + \sigma$, where σ is a detuning parameter and applying the method of multiple scales [33], we obtain the slow flow of the oscillator (2)

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t} &= Au + Sv - (Bu + Cv)(u^2 + v^2)\\ &\quad - H_1 v(v^2 - 3u^2) - H_2 u(u^2 - 3v^2),\\ \frac{\mathrm{d}v}{\mathrm{d}t} &= -Su + Av - (Bv - Cu)(u^2 + v^2)\\ &\quad + H_1 u(u^2 - 3v^2) - H_2 v(v^2 - 3u^2),\end{aligned} \tag{3}$$

where $A = \frac{\alpha}{2}, B = \frac{2\beta c}{\omega^2}, C = \frac{80c^2}{3\omega^3} + \frac{\beta^2}{6\omega}, S = \frac{2\sigma}{\omega} + \frac{h^2\omega}{192}, H_1 = \frac{h\beta^2}{72\omega} - \frac{20hc^2}{9\omega^3}, H_2 = \frac{h\beta c}{9\omega^2}, u = r\cos\theta, v = -r\sin\theta$ with r and θ are, respectively, the amplitude and the phase of the slow flow system (3) in its polar form [detail on the derivation of the slow flow (3) is given in [28]]. Note that this slow flow (3) is equivalent in its polar form to the normal form (1) and then investigating the dynamic of the normal form or the slow flow is the same.

The two heteroclinic connections occurring in the slow flow (3) (the *square* connection formed around the trivial equilibrium and the *clover* one surrounding both trivial and nontrivial equilibria) exhibit the $\mathbf{Z}_4$-symmetry. By applying the collision criterion between the slow flow limit cycle and the saddles, analytical approximation of the *square* connection was obtained [28] and acceptable agreement with numerical simulation was shown. However, the *clover* connection has not been considered because the slow flow limit cycle required for applying the collision criterion has to be approximated near outside the connection which is not an easy task.

In the present study, we take advantage of the unperturbed Hamiltonian system of the slow flow (3) and the zero-order approximation of a heteroclinic connection to capture the two heteroclinic connections simultaneously. To obtain such a zero-order approximation, we first determine the condition under which the system (3) is a Hamiltonian system. To this end, assume

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{\partial H}{\partial v} \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\partial H}{\partial u}, \tag{4}$$

where H is a Hamiltonian function. Since

$$\frac{\partial}{\partial u}\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right) + \frac{\partial}{\partial v}\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right) = 2A - 4B(u^2 + v^2) = 0,$$

for all $u,\ v \in \mathbf{R}$, we have the condition

$$A = B = 0. \tag{5}$$

To simplify the subsequent calculations and due to the $\mathbf{Z}_4$-symmetry of (3), we assume that the heteroclinic connections pass through the four equilibria $E_{\pm,\pm} = \left(\pm\frac{r_0}{\sqrt{2}}, \pm\frac{r_0}{\sqrt{2}}\right)$ which are at a distance r_0 from the origin. Substituting the equilibria into (3) yields

$$H_2 = 0 \quad \text{and} \quad r_0 = \sqrt{\frac{S}{C - H_1}}. \tag{6}$$

To make sure that r_0 in (6) is real and the value inside a square root arisen in subsequent calculation is always positive, we impose the following condition

$$S > 0 \quad \text{and} \quad C > H_1. \tag{7}$$

Integrating the right-hand side of (3) together with the conditions (5) and (6), we obtain the following solution of (3), which passes through the equilibria $E_{\pm,\pm}$:

$$\begin{aligned}&C(u^2 + v^2)^2 - 2S(u^2 + v^2)\\ &\quad + H_1(u^4 - 6u^2v^2 + v^4) + \frac{S^2}{C - H_1} = 0.\end{aligned} \tag{8}$$

In [29,30], a nonlinear time transformation technique was introduced to study limit cycles of strongly nonlinear oscillators and delay differential equations. This technique was extended to study homoclinic/heteroclinic orbits in [31] and coherent structures in [32]. Here, we apply this technique to approximate heteroclinic bifurcations in the oscillator (2) near the 1:4 resonance. To this end, we define the nonlinear time transformation as

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \Phi(\varphi) \quad \text{where} \quad \Phi(\varphi) = \Phi(\varphi + 2\pi). \tag{9}$$

To find an analytical expression of the zero-order heteroclinic orbit passing through the two equilibria $E_{\pm,+} = \left(\pm\frac{r_0}{\sqrt{2}}, \frac{r_0}{\sqrt{2}}\right)$, we assume

$$u = \frac{r_0}{\sqrt{2}}\cos\varphi, \tag{10}$$

so that the heteroclinic orbit is at $E_{+,+}$ and $E_{-,+}$ when $\varphi = 0$ and $\varphi = \pi$, respectively. Substituting (10) into (8), we obtain two solutions in v^2 as

$$v_\pm = \frac{r_0}{2}\sqrt{(c_\pm - d_\pm \cos 2\varphi)}, \tag{11a}$$

$$c_\pm = \frac{3C - H_1 + \sqrt{2}k_\pm}{C + H_1}, \quad d_\pm = \frac{C - 3H_1 + \sqrt{2}k_\pm}{C + H_1},$$
$$k_\pm = \pm 2\sqrt{-H_1(C - H_1)}. \tag{11b}$$

Condition (7) guarantees the value inside the square root of (11a) to be positive for $\varphi \in [0, 2\pi]$. Equations (10), (11) provide the zero-order approximation of the heteroclinic connection. In addition to condition (7), it follows from (11b) that the condition $H_1 < 0$ has to be satisfied as well.

For this Hamiltonian system, a *square* and a *clover* heteroclinic connections are given by (u, v_-) and (u, v_+), respectively, for $\varphi \in (0, \pi)$. For the *square* connection, it follows from (3) and (9)–(11) that

$$\Phi_- = \frac{\mathrm{d}u/\mathrm{d}t}{\mathrm{d}u/\mathrm{d}\varphi} = k_- r_0 v_- \sin\varphi. \tag{12}$$

Similarly, for the *clover* connection, we have

$$\Phi_+ = k_+ r_0 v_+ \sin\varphi. \tag{13}$$

At the equilibria $E_{\pm,+}$ where $\varphi \in \{0, \pi\}$, we note that

$$\Phi_\pm(0) = \Phi_\pm(\pi) = 0, \tag{14}$$

which is the condition in (5) of [31].

3 *Square* and *Clover* heteroclinic connections

To capture both *square/clover* heteroclinic bifurcation, we consider a perturbation of the Hamiltonian system discussed above by assuming $A = \varepsilon A_1$, $B = \varepsilon B_1$ and $H_2 = \varepsilon H_{21}$ where ε is a small perturbation parameter. The heteroclinic orbits $(u, v_\pm)$ obtained from (10) and (11) can be regarded as the zero-order approximation of the *square/clover* connections. For simplicity of the symbols, we drop the "±" sign and bear in mind that "−" refers to a *square* connection while "+" a *clover* connection. From (6) and (10)–(12), the zero-order approximation is expressed explicitly as

$$\begin{cases} r_0 = \sqrt{\dfrac{S}{C - H_1}}, & (15a)\\ u_0 = \dfrac{r_0}{\sqrt{2}}\cos\varphi, & (15b)\\ v_0 = \dfrac{r_0}{2}\sqrt{c - d\cos 2\varphi}, & (15c)\\ \Phi_0 = k r_0 v_0 \sin\varphi, & (15d)\end{cases}$$

where c, d and k are given in (11b). Perturbation of this zero-order approximation allows one to assume that the first-order approximation of u, v and Φ is given by

$$\begin{cases} u = u_0 + \varepsilon b v_0 + O(\varepsilon^2),\\ v = v_0 + \varepsilon v_1 + O(\varepsilon^2),\\ \Phi = \Phi_0 + \varepsilon \Phi_1 + O(\varepsilon^2),\end{cases} \tag{16}$$

where $b \in \mathbf{R}$ and v_1, Φ_1 are functions of φ. Substituting (16) into (3) and equating the coefficients of ε lead to the following equations

$$\begin{aligned}&[C(r_0^2 - u_0^2 - 3v_0^2) + H_1(-r_0^2 + 3u_0^2 - 3v_0^2)]v_1\\ &+ A_1 u_0 - B_1 u_0(u_0^2 + v_0^2)\\ &+ H_{21}u_0(3v_0^2 - u_0^2) + b[2(3H_1 - C)u_0 v_0^2\\ &- \Phi_0\frac{\mathrm{d}v_0}{\mathrm{d}\varphi}] - \Phi_1\frac{\mathrm{d}u_0}{\mathrm{d}\varphi} = 0,\end{aligned} \tag{17a}$$

$$\begin{aligned}&-\Phi_0\frac{\mathrm{d}v_1}{\mathrm{d}\varphi} + 2(C - 3H_1)u_0 v_0 v_1\\ &+ A_1 v_0 - B_1 v_0(u_0^2 + v_0^2) + H_{21}v_0(3u_0^2 - v_0^2)\\ &+ b v_0[C(-r_0^2 + 3u_0^2 + v_0^2)\\ &+ H_1(r_0^2 + 3u_0^2 - 3v_0^2)] - \Phi_1\frac{\mathrm{d}v_0}{\mathrm{d}\varphi} = 0.\end{aligned} \tag{17b}$$

Substituting (15a)–(15d) into (17a)–(17b) and setting $\varphi = 0$, we obtain

$$\begin{cases} v_1(0) = -\dfrac{b r_0}{\sqrt{2}}, & (18a)\\ A_1 + r_0^2(4H_1 b + H_{21} - B_1) = 0. & (18b)\end{cases}$$

We can also obtain the above equations if we set $\varphi = \pi$. To solve (17a) and (17b), we consider a change in variable as

$$v_1 = w - b u_0. \tag{19}$$

It follows from (18a) and (19) that

$$w(0) = 0. \tag{20}$$

Substituting (19) into (17a) and (17b) and eliminating Φ_1 between the two equations yields

$$\begin{aligned}&-k\frac{\mathrm{d}}{\mathrm{d}\varphi}(w v_0 \sin^2\varphi) + \frac{(2v_0^2 - r_0^2)\sin\varphi}{\sqrt{2}(C + H_1)r_0^3 v_0}\\ &\Big[2\sqrt{2}(C - H_1)(A_1 - B_1 r_0^2)(2v_0^2 - r_0^2)\\ &- k(A_1 r_0^2 + 4A_1 v_0^2 - 4B_1 r_0^2 v_0^2)\Big] = 0.\end{aligned} \tag{21}$$

Integrating (21) with respect to φ, we obtain

$$-kwv_0\sin^2\varphi = \frac{1}{(C+H_1)r_0^3}\left\{\begin{array}{l}4(B_1r_0^2-A_1)(2C-2H_1-\sqrt{2}k)\Gamma_3+r_0^2\big[\,A_1\,(8C\\ -8H_1-\sqrt{2}k\big)-2B_1r_0^2(4C-4H_1-\sqrt{2}k)\big]\,\Gamma_1\\ +r_0^4[2B_1r_0^2(C-H_1)-A_1(2C-2H_1+\frac{k}{\sqrt{2}})]\Gamma_{-1}\end{array}\right\}, \qquad (22)$$

$$\begin{aligned}\Gamma_3 &= \int_0^{\varphi} v_0^3\sin\varphi_1 \mathrm{d}\varphi_1\\ &= \left(\frac{r_0}{2}\sqrt{\frac{d}{2}}\right)^3\Big[2(a-1)^{3/2}+3a\sqrt{a-1}\\ &\quad +3a^2\tan^{-1}\left(\frac{1}{\sqrt{a-1}}\right)\\ &\quad -2\cos\varphi(a-\cos^2\varphi)^{3/2}-3a\cos\varphi\sqrt{a-\cos^2\varphi}\\ &\quad -3a^2\tan^{-1}\left(\frac{\cos\varphi}{\sqrt{a-\cos^2\varphi}}\right)\Big],\\ \Gamma_1 &= \int_0^{\varphi} v_0\sin\varphi_1\mathrm{d}\varphi_1 = \frac{r_0}{2}\sqrt{\frac{d}{2}}\Big[\sqrt{a-1}\\ &\quad +a\tan^{-1}\left(\frac{1}{\sqrt{a-1}}\right)-\cos\varphi\sqrt{a-\cos^2\varphi}\\ &\quad -a\tan^{-1}\left(\frac{\cos\varphi}{\sqrt{a-\cos^2\varphi}}\right)\Big],\\ \Gamma_{-1} &= \int_0^{\varphi}\frac{\sin\varphi_1}{v_0}\mathrm{d}\varphi_1 = \frac{\sqrt{2}}{r_0\sqrt{d}}\Big[\tan^{-1}\left(\frac{1}{\sqrt{a-1}}\right)\\ &\quad -\tan^{-1}\left(\frac{\cos\varphi}{\sqrt{a-\cos^2\varphi}}\right)\Big],\end{aligned}$$

$$a = \frac{c+d}{2d} = \frac{2(C-H_1)-\sqrt{2}k}{C+H_1}. \qquad (23)$$

In (23), for a *square* (*clover*, respectively) connection, $a = a_-$ ($a = a_+$, resp.) with c, d and k replaced by c_-, d_- and k_- (c_+, d_+ and k_+, resp.). To find the relationship between A_1 and B_1 so that a *square* or *clover* heteroclinic connection persists, we set $\varphi = \pi$ in (22) and obtain

$$\begin{aligned}y := {}& 2(C-H_1)\Big[\sqrt{2}(C-H_1)\\ &+k]\,[SB_1-(C+H_1)A_1]\tan^{-1}\left(\frac{1}{\sqrt{a-1}}\right)\\ &+SB_1k\sqrt{C+H_1}\sqrt{C-3H_1+\sqrt{2}k}=0. \quad (24)\end{aligned}$$

It is interesting to note from (24) that when A in (3) varies from zero, the persistence of *square* or *clover* heteroclinic connections depends on B only. To be consistent, we fix the parameter values as in [28], namely we set $\omega_0 = 1$, $\alpha = 0.01$, $\beta = 0.2$ and $c = 1$ for the system parameters $A = \varepsilon A_1$, $B = \varepsilon B_1$, C, H_1, $H_2 = \varepsilon H_{21}$ and S in (3).

Figure 1 shows in the forcing parameter plane the analytical prediction (solid lines) of the *square* (respectively, *clover*) heteroclinic connection, plotted by the curve labelled H_s (respectively, H_c) given by the criterion $y_-(\omega, h) = 0$ (respectively, $y_+(\omega, h) = 0$) obtained from Eq. (24). These analytical predictions are compared to the results obtained by numerical simulation using Runge–Kutta method (dots) showing an excellent agreement for the two heteroclinic connection curves. The analytical curves, labelled SN, denote the saddle-node bifurcation delimiting the region where the 1:4 subharmonic cycles exist.

Figure 2 illustrates the examples of phase portraits of the slow flow (3) for the values of ω picked from Fig. 1 and for $h = 0.2$. As the forcing frequency varies,

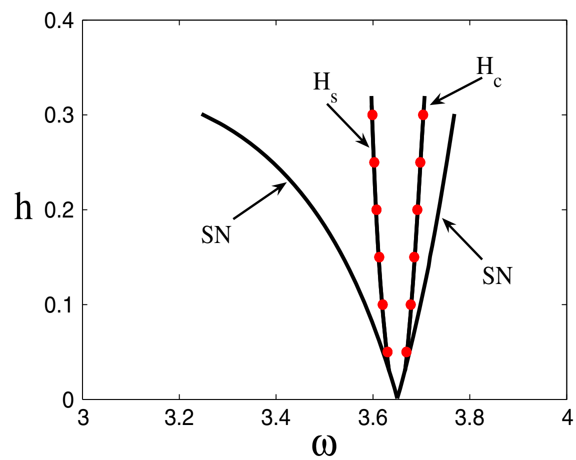

Fig. 1 Bifurcation curves near the 1:4 resonance for $\alpha = 0.01$, $\beta = 0.2$. H_s *square* heteroclinic bifurcation, H_c *clover* heteroclinic bifurcation, SN saddle-node bifurcation curves. Analytical approximation (*solid lines*), numerical simulation (*dots*)

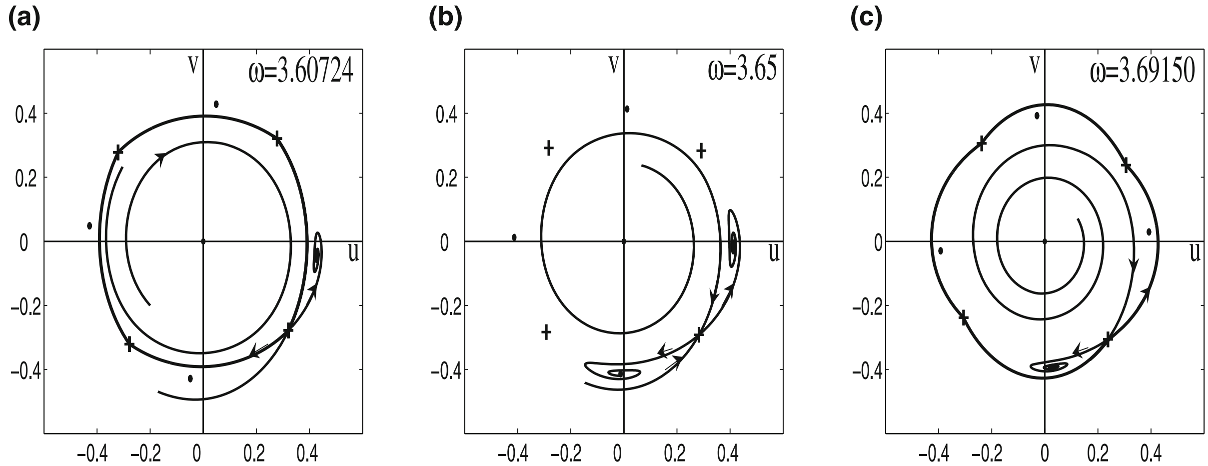

Fig. 2 Examples of phase portraits of the slow flow at different frequencies picked from Fig. 1 and for $\alpha = 0.01$, $\beta = 0.2$, $h = 0.2$. **a** *Square* connection, **b** *synchronization*, **c** *clover* connection

the stable slow flow limit cycle born by Hopf bifurcation (not shown here) approaches the saddles and disappears via a *square* connection, as shown in subfigure (a) for $\omega = 3.60724$. As ω increases, the *clover* connection occurs for $\omega = 3.69150$ as shown in subfigure (c) (for more details on the bifurcation, see [28]). The frequency-locking domain is delimited by the heteroclinic bifurcation curves H_s and H_c in which the response of the system follows the subharmonic frequency [see subfigure (b) for $\omega = 3.65$].

In order to validate the results obtained by the nonlinear transformation method, we analyze the results for larger values of the damping coefficients: $\alpha = 0.1$, $\beta = 0.8$. For these values of parameters, Fig. 3 shows in the parameter plane (h, ω) the analytical prediction of the heteroclinic connections, H_s, H_c, as given by the criterion $y_-(\omega, h) = 0$, $y_+(\omega, h) = 0$, respectively (solid lines). These bifurcation curves are plotted for values of parameters lying inside the saddle-node region where heteroclinic connections occur. The results obtained by numerical simulation using Runge–Kutta method (dots) are also plotted showing a good agreement inside the saddle-node region.

Examples of phase portraits of the slow flow (3) are shown in Fig. 4 for $h = 0.6$. The *square* and the *clover* heteroclinic connections are shown in subfigures (a) and (c) for $\omega = 2.7841$ and $\omega = 3.2114$, respectively.

It is worth noticing that for small damping $\alpha = 0.01$ and $\beta = 0.2$, the analytical finding predicts well the heteroclinic bifurcations (Fig. 1). Instead, for a relatively larger value of damping, $\alpha = 0.1$ and $\beta = 0.8$,

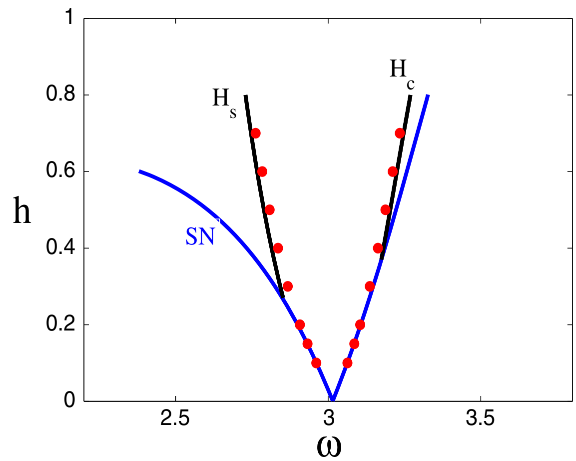

Fig. 3 Bifurcation curves near the 1:4 resonance for $\alpha = 0.1$, $\beta = 0.8$. H_s *square* heteroclinic bifurcation, H_c *clover* heteroclinic bifurcation, *SN* saddle-node bifurcation curves. Analytical approximation (*solid lines*), numerical simulation (*dots*)

the analytical predictions match the numerical simulation for large values of the amplitude excitation h (Fig. 5). For small values of the amplitude excitation, on the other hand, the analytical approximation lies outside the region of existence of the saddle-node cycle (see dashed lines in Fig. 5). The minimum value of h for which the method is valid would correspond to the intersection locations between the heteroclinic and the saddle-node curves. As depicted in Figs. 1 and 3, these minimum values of h increase as α increases.

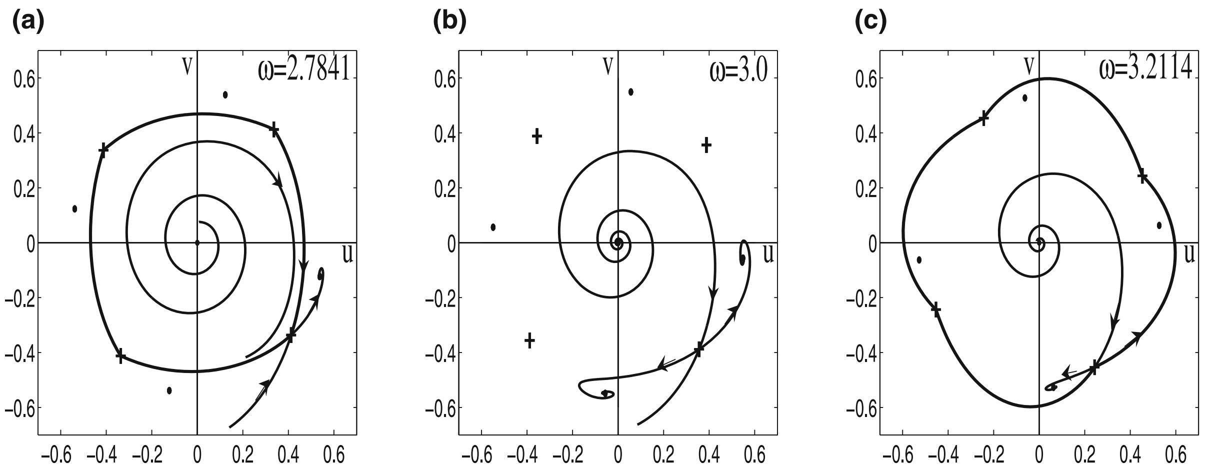

Fig. 4 Examples of phase portraits of the slow flow at different frequencies picked from Fig. 3 and for $\alpha = 0.1$, $\beta = 0.8$, $h = 0.6$. **a** *Square* connection, **b** *synchronization*, **c** *clover* connection

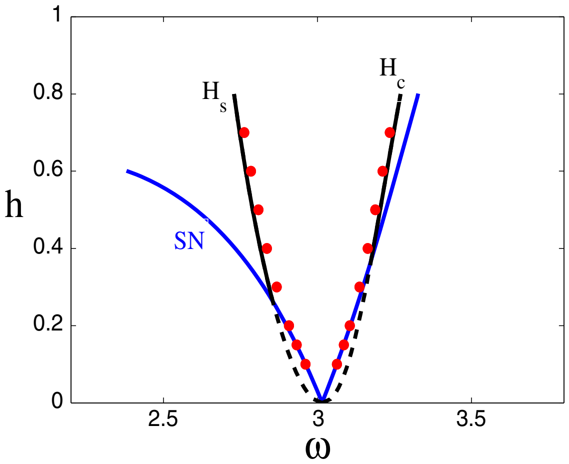

Fig. 5 Bifurcation curves near the 1:4 resonance for $\alpha = 0.1$, $\beta = 0.8$. *SN* Saddle-node bifurcation curves, H_s *square* heteroclinic bifurcation, H_c *clover* heteroclinic bifurcation. Analytical approximation (*solid* and *dashed lines*), numerical simulation (*dots*)

4 Conclusion

In this paper, we have applied a nonlinear time transformation method to obtain analytical prediction of both *square* and *clover* heteroclinic connections in the 1:4 resonance problem considering as application of a self-excited parametric nonlinear oscillator. The method is principally based on perturbing the zero-order approximation of the heteroclinic connection in the slow flow and determines conditions on parameters under which the perturbed heteroclinic connection persists. It was shown that for small damping, the nonlinear transformation method offers a powerful analytical tool for predicting heteroclinic bifurcations in the problem of stability loss of self-oscillations near the 1:4 resonance.

The approximation of the *square* heteroclinic connection obtained by the present method improves substantially the approximation given by the collision criterion [28]. Moreover, the nonlinear time transformation method is able to predict also the *clover* heteroclinic connection, which was not done using the collision criterion.

The original results of the present study indicate that the nonlinear transformation method can be used to predict analytically heteroclinic bifurcation, which are of importance in the problem of synchronization for systems of globally coupled phase oscillators and in determining coherent structures in physics and biology.

Acknowledgments This research was supported by the Strategic Research Grant 7002814 of the City University of Hong Kong.

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