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Control of a Delayed Limit Cycle Using the Tilt Angle of a Fast Excitation

Si Mohamed Sah and Mohamed Belhaq

Abstract

We investigate the tilting effect of a fast excitation on self-excited vibrations in a delayed van der Pol pendulum. Specifically, we analyze the simultaneous effects of fast excitation and time delay in controlling self-excited oscillations taking into account the influence of the incline angle of the fast excitation. We use an averaging technique to reduce the original oscillator to a slow flow system. Analysis of stationary solutions of this slow flow provides analytical approximations of regions in parameter space where self-excited vibrations can be controlled. We have shown that in the case where the delay and the fast excitation are both imposed in the system, the incline of the fast excitation can be an alternative to control self-excited vibrations.

Keywords

control, delay, fast-excitation, perturbation, self-excitation

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1. Introduction

The study of nontrivial effects of a fast harmonic excitation (FHE) on the dynamic of mechanical systems has received great attention in the last decade. The method of direct partition of motion (DPM) (Blekhman, 2000) is used for analyzing such effects. This technique, based on splitting the dynamic into fast and slow motions, provides an approximation for the small fast dynamic and the main equation governing the slow motion. Previous works used this technique to analyze properties and dynamics of some mechanical systems under FHE (Tcherniak and Thomsen 1998, Thomsen, 1999, 2002; Chatterjee et al., 2003; Fidlin and Thomsen, 2004; Thomsen, 2005). The effect of a FHE on the suppression of limit cycle in a van der Pol pendulum has been examined by Bourkha and Belhaq (2007). It was shown that the limit cycle can be eliminated by a horizontal FHE for a certain threshold of the frequency and persists in the vertical FHE case for all values of the fast excitation. However, Sah and Belhaq (2008) and Belhaq and Sah (2008a) showed that the vertical FHE may eliminate the self-excited vibrations when a time delay is added. Recently, Belhaq and coworkers (Belhaq and Sah, 2008b; Belhaq and Fahsi, 2008; Fahsi et al., 2009; Fahsi and Belhaq, 2009) investigated the effect of a fast excitation on the hysteresis phenomenon and shown that the fast frequency can be chosen so as to completely eliminate the hysteresis loop.

In this paper, we examine the tilting effect of a fast excitation on the control of self-excited vibrations in a delayed van der Pol pendulum. This work was motivated by the important issue of controlling self-excited vibrations in some mechanical applications in which a delay and a fast excitation are imposed. In this case, using the tilt angle of the fast excitation can be an alternative for the control purpose. This study extends previous works (Sah and Belhaq, 2008; Belhaq and Sah, 2008a,b) in which the effect of horizontal and vertical excitations were investigated separately. In other words, these works analyzed the cases where the incline angle of the fast excitation is either vertical (Sah and Belhaq, 2008) or horizontal (Belhaq and Sah, 2008b). In contrast, the present work allows the angle of the fast excitation to vary between the horizontal and the vertical lines.

In Section 2, we apply the DPM technique to derive the main autonomous equation governing the slow dynamic of the oscillator in the tilting excitation case. The method of averaging is then performed on the slow dynamic to

Laboratory of Mechanics, University Hassan II, Casablanca, Morocco

Corresponding Author:
Si Mohamed Sah, Laboratory of Mechanics, University Hassan II, Casablanca, Morocco
Email: simonhamedsah@yahoo.fr

obtain a slow flow which is analyzed for equilibria and limit cycle, providing analytical predictions for the control of self-excited vibrations. Section 3 is devoted to the limiting cases of horizontal and vertical excitations. We conclude in Section 4.

2. Tilted Excitation

Vibrations of a pendulum with time delay subjected to a tilted fast forcing and to a self-excitation can be described in the following nondimensional equation

$$\frac{d^2x}{dt^2} - (\alpha - \beta x^2)\frac{dx}{dt} + \sin x = a\Omega^2 \cos(x-\theta) \cos\Omega t + \lambda x(t-T), \tag{1}$$

where the damping parameters α and β are assumed to be small, a and Ω are the amplitude and the frequency of the fast excitation, respectively, θ measures the inclination angle of the excitation with the horizontal, and the parameters λ and T are the amplitude of the delay and the delay period, respectively. Equation 1 has relevance to regenerative effect in high-speed milling. High-speed milling can induce a rapid parametric excitation and milling can generate self-oscillations. We focus our analysis on small vibrations around $x = 0$ by expanding in Taylor's series up to the third-order the terms $\sin x \simeq x - \delta x^3$ and $\cos x \simeq 1 - \gamma x^2$ where the coefficients $\delta = 1/6$ and $\gamma = 1/2$. Equation 1 becomes

$$\frac{d^2x}{dt^2} - (\alpha - \beta x^2)\frac{dx}{dt} + (x - \delta x^3) = a\Omega^2((1-\gamma x^2)C + (x-\delta x^3)S)\cos\Omega t + \lambda x(t-T), \tag{2}$$

where it is assumed that $C = \cos\theta$ and $S = \sin\theta$.

2.1. Slow Dynamic

We implement the method of DPM by introducing a fast time $T_0 \equiv \Omega t$ and a slow time $T_1 \equiv t$, and we split up $x(t)$ into a slow part $z(T_1)$ and a fast part $\varepsilon\varphi(T_0, T_1)$ as follows

$$x(t) = z(T_1) + \varepsilon\varphi(T_0, T_1) \tag{3}$$

and

$$x(t-T) = z(T_1 - T) + \varepsilon\varphi(T_0 - \Omega T, T_1 - T), \tag{4}$$

where z describes slow motions at the time-scale of oscillations of the pendulum, and $\varepsilon\varphi$ stands for an overlay of the fast motions. In equations 3 and 4, ε indicates that $\varepsilon\varphi$ is small compared with z. Since Ω is considered as a large parameter we choose $\varepsilon \equiv \Omega^{-1}$, for convenience. The fast part $\varepsilon\varphi$ and its derivatives are assumed to have a zero T_0-average, so that $\langle x(t)\rangle = z(T_1)$ and $\langle x(t-T)\rangle = z(T_1 - T)$ where $\langle\rangle \equiv (1/2\pi)\int_0^{2\pi}()\,dT_0$ defines time-averaging operator over one period of the fast excitation with the slow time T_1 fixed.

Inserting equations 3 and 4 into equation 2 and introducing

$$D_i^j \equiv \frac{\partial^j}{\partial^j T_i}$$

yields

$$\begin{aligned} &D_1^2 z + 2D_0D_1\varphi - \alpha(D_1 z + D_0\varphi) + \beta(z^2 D_1 z + z^2 D_0\varphi) \\ &\quad + z - \delta z^3 = \varepsilon^{-1}(a\Omega)(z + \varepsilon\varphi - \delta(z^3 + 3\varepsilon z^2\varphi)) \\ &\quad S\cos T_0 + \varepsilon^{-1}(a\Omega)(1 - \gamma(z^2\cos T_0 + 2\varepsilon z\varphi)) \\ &\quad C\cos T_0 + \lambda z(T) \end{aligned} \tag{5}$$

Averaging equation 5 leads to

$$D_1^2 z - \alpha D_1 z + \beta z^2 D_1 z + z - \delta z^3 = -2\gamma(a\Omega)z\langle\varphi\cos T_0\langle C + (a\Omega)(1 - 3\delta z^2)\langle\varphi\cos T_0\rangle S + \lambda z(T_1 - T). \tag{6}$$

Subtracting equation 6 from equation 5, an approximate expression for $\varepsilon\varphi$ is obtained by considering only the dominant terms of order ε^{-1} as

$$D_0^2\varphi = a\Omega((1-\gamma z^2)C + (z - \delta z^3)S)\cos T_0, \tag{7}$$

where it is assumed that $a\Omega = O(\varepsilon^0)$. The stationary solution to the first-order for φ is written as

$$\varphi = -a\Omega((1-\gamma z^2)C + (z-\delta z^3)S)\cos T_0. \tag{8}$$

Retaining the dominant terms of order ε^0 in equation 6, inserting φ from equation 8 and using that $\langle\cos^2 T_0\rangle = 1/2$ gives the following autonomous equation governing the slow dynamic of the motion

$$\begin{aligned} &D_1^2 z - (\alpha - \beta z^2)D_1 z + (1 - (a\Omega)^2(\gamma C^2 - \frac{S^2}{2})) \\ &CSz - \frac{3}{2}(a\Omega)^2(\gamma + \delta)CSz^2 - (\delta(1 + 2(a\Omega)^2 S^2)) \\ &\quad - \gamma^2(a\Omega)^2 C^2)z^3 = -\frac{(a\Omega)^2}{2}CS + \lambda z(t-T). \end{aligned} \tag{9}$$

2.2. Averaging and Slow Flow

We use the averaging method (Nayfeh and Mook, 1979; Rand, 2005) by introducing a small parameter μ and we scale parameters $\alpha = \mu\tilde{\alpha}$, $\beta = \mu\tilde{\beta}$, $\gamma = \mu\tilde{\gamma}$, $\delta = \mu\tilde{\delta}$ and $\lambda = \mu\tilde{\lambda}$. Then, equation 9 reads

$$\begin{aligned} &\ddot{z} - \mu(\tilde{\alpha} - \tilde{\beta}z^2)\dot{z} + \omega_0^2 z - \frac{3}{2}(a\Omega)^2\mu(\tilde{\gamma} + \tilde{\delta})CSz^2 \\ &\quad - ((\mu\tilde{\delta}(1 + 2(a\Omega)^2 S^2)) \\ &\quad - \mu^2\tilde{\gamma}^2(a\Omega)^2 C^2)z^3 = \mu\tilde{\lambda}z(t-T) - \frac{(a\Omega)^2}{2}CS, \end{aligned} \tag{10}$$

where $\dot{z} = dz/dt$ and $\omega_0^2 = 1 + ((a\Omega)^2/2)(S^2 - 2\gamma C^2)$. In the case $\mu = 0$, equation 10 reduces to

$$\ddot{z}(t) + \omega_0^2 z(t) = K, \tag{11}$$

where $K = -((a\Omega)^2/2)CS$ and with the solution

$$z(t) = R\cos(\omega_0 t + \varphi) + \frac{K}{\omega_0^2}, \quad \dot{z}(t) = -R\omega_0 \sin(\omega_0 t + \varphi). \tag{12}$$

For $\mu > 0$, a solution is sought in the form of equation 12 where R and φ are time dependent. Variations of parameters gives the following equations on $R(t)$ and $\varphi(t)$:

$$\begin{aligned}\dot{R}(t) = &-\frac{\mu}{\omega_0}\sin(\omega_0 t + \varphi)F_1(R\cos(\omega_0 t + \varphi), \\ &- \omega_0 R\sin(\omega_0 t + \varphi), t) - \frac{\mu^2}{\omega_0}\sin(\omega_0 t + \varphi) \\ &F_2(R\cos(\omega_0 t + \varphi), -\omega_0 R\sin(\omega_0 t + \varphi), t),\end{aligned} \tag{13}$$

$$\begin{aligned}\dot{\varphi}(t) = &-\frac{\mu}{\omega_0 R}\cos(\omega_0 t + \varphi)F_1(R\cos(\omega_0 t + \varphi), \\ &- \omega_0 R\sin(\omega_0 t + \varphi), t) - \frac{\mu^2}{\omega_0 R}\cos(\omega_0 t + \varphi) \\ &F_2(R\cos(\omega_0 t + \varphi), -\omega_0 R\sin(\omega_0 t + \varphi), t),\end{aligned} \tag{14}$$

Where

$$\begin{aligned}F_1(z, \dot{z}, t) = &(\tilde{\alpha} - \tilde{\beta}z^2)\dot{z} + \frac{3}{2}(a\Omega)^2(\tilde{\gamma} + \tilde{\delta})CSz^2 \\ &+ \tilde{\delta}(1 + 2(a\Omega)^2 S^2)z^3 + \tilde{\lambda}z(t - T)\end{aligned} \tag{15}$$

and

$$F_2(z, \dot{z}, t) = -(a\Omega)^2\tilde{\gamma}^2 C^2 z^3 \tag{16}$$

with $z(t)$ is given by equation 12. Averaging for small μ and replacing the right-hand sides of equations 13 and 14 by their averages over one 2π-period, since equation 10 is autonomous, we obtain

$$\begin{aligned}\dot{R} \approx &-\frac{\mu}{\omega_0}\frac{1}{2\pi}\int_0^{2\pi}\sin(\omega_0 t + \varphi)F_1\,dt - \frac{\mu^2}{\omega_0} \\ &\frac{1}{2\pi}\int_0^{2\pi}\sin(\omega_0 t + \varphi)F_2\,dt\end{aligned} \tag{17}$$

$$\begin{aligned}\dot{\varphi} \approx &-\frac{\mu}{\omega_0 R}\frac{1}{2\pi}\int_0^{2\pi}\cos(\omega_0 t + \varphi)F_1\,dt - \frac{\mu^2}{\omega_0 R} \\ &\frac{1}{2\pi}\int_0^{2\pi}\cos(\omega_0 t + \varphi)F_2\,dt\end{aligned} \tag{18}$$

in which

$$\begin{aligned}F_1 = &-(\tilde{\alpha} - \tilde{\beta}(R\cos(\omega_0 t + \varphi) + \frac{K}{\omega_0^2})^2)\,\omega_0 R\sin(\omega_0 t + \varphi) \\ &+ \tilde{\delta}(1 + 2(a\Omega)^2 S^2)(R\cos(\omega_0 t + \varphi) + \frac{K}{\omega_0^2})^3 \\ &+ \frac{3}{2}(a\Omega)^2(\tilde{\gamma} + \tilde{\delta})\,CS(R\cos(\omega_0 t + \varphi) + \frac{K}{\omega_0^2})^2 \\ &+ \tilde{\lambda}\tilde{R}\cos(\omega_0 t - \omega_0 T + \tilde{\varphi})\end{aligned} \tag{19}$$

and

$$F_2 = -(a\Omega)^2\tilde{\gamma}^2 C^2 R^3\cos(\omega_0 t + \varphi)^3 \tag{20}$$

with $\tilde{R} = R(t - T)$ and $\tilde{\varphi} = \varphi(t - T)$. Evaluating the integrals in equations 17 and 18 yields

$$\dot{R} = \mu\left(\frac{\tilde{\alpha}}{2}R - \frac{\tilde{\beta}K^2}{2\omega_0^4}R - \frac{\tilde{\beta}}{8}R^3 - \frac{\tilde{\lambda}}{2\omega_0}\tilde{R}\sin(\omega_0 T - \tilde{\varphi} + \varphi)\right) \tag{21}$$

$$\begin{aligned}\dot{\varphi} = \mu\Bigg(&-\tilde{\delta}(1 + 2(a\Omega)^2 S^2)\left(\frac{3R^2}{8\omega_0} + \frac{3K^2}{2\omega_0^3}\right) \\ &-\frac{3(a\Omega)^2}{2\omega_0^3}(\tilde{\gamma} + \tilde{\delta})CSK - \frac{\tilde{\lambda}\tilde{R}}{2\omega_0 R}\cos(\omega_0 T - \tilde{\varphi} + \varphi)\Bigg) \\ &+ \mu^2\left((\gamma^2(a\Omega)^2 C^2\left(\frac{3}{8\omega_0}R^2 + \frac{3K}{2\omega_0^5}\right)\right).\end{aligned} \tag{22}$$

Equations 21 and 22 show that $\dot{R}$ and $\dot{\varphi}$ are $O(\mu)$. We now expand in Taylor's series $\tilde{R}$ and $\tilde{\varphi}$ as follows

$$\tilde{R} = R(t - T) = R(t) - T\dot{R}(t) + \frac{1}{2}T^2\ddot{R}(t) + \cdots, \tag{23}$$

$$\tilde{\varphi} = \varphi(t - T) = \varphi(t) - T\dot{\varphi}(t) + \frac{1}{2}T^2\ddot{\varphi}(t) + \cdots. \tag{24}$$

Then, we can replace $\tilde{R}$ and $\tilde{\varphi}$ by $R(t)$ and $\varphi(t)$ in equations 21 and 22 since $\dot{R}$, $\dot{\varphi}$, $\ddot{R}$ and $\ddot{\varphi}$ are of $O(\mu)$ and $O(\mu^2)$, respectively (Wirkus and Rand, 2002). This approximation reduces the infinite-dimensional problem into a finite-dimensional one by assuming that μT is small. After substituting the above approximation into equations 21 and 22, we obtain the following slow flow of the slow dynamic of motion

$$\dot{R} = \left(\frac{\alpha}{2} - \frac{\beta}{2\omega_0^4}K^2 R - \frac{\lambda}{2\omega_0}\sin\omega_0 T\right)R - \frac{\beta}{8}R^3, \tag{25}$$

$$\dot{\varphi} = -\frac{3K^2}{2\omega_0^3}\delta(1+2(a\Omega)^2S^2) - \frac{3(a\Omega)^2}{2\omega_0^3}(\gamma+\delta)CSK + \frac{3K}{2\omega_0^5}\gamma^2(a\Omega)^2C^2 - \frac{\lambda}{2\omega_0}\cos\omega_0 T + \frac{3}{8\omega_0}(\gamma^2(a\Omega)^2C^2 - \delta(1+2(a\Omega)^2S^2))R^2. \quad (26)$$

2.3. Self-excited Oscillations

An equilibrium in equations 25 and 26 corresponds to a periodic motion in the system of equation 9. Equilibria are obtained by setting $\dot{R} = \dot{\varphi} = 0$ in equations 25 and 26. This leads to

$$R = 0, \quad R = \sqrt{\frac{8}{\beta}\left(\frac{\alpha}{2} - \frac{\lambda}{2\omega_0}\sin\omega_0 T - \frac{\beta K^2}{2\omega_0^4}\right)}, \quad (27)$$

where

$$\omega_0 = \sqrt{1 + \frac{(a\Omega)^2}{2}(S^2 - 2\gamma C^2)}.$$

The solution

$$R = \sqrt{\frac{8}{\beta}\left(\frac{\alpha}{2} - \frac{\lambda}{2\omega_0}\sin\omega_0 T - \frac{\beta K^2}{2\omega_0^4}\right)}$$

corresponding to a periodic motion (limit cycle) must be real. This leads to the condition

$$\alpha - \frac{\lambda}{\omega_0}\sin\omega_0 T - \frac{\beta K^2}{\omega_0^4} \geq 0. \quad (28)$$

Solving the above inequality for T provides the two following conditions, denoted by (C1), corresponding to the birth of the limit cycle

$$T < \frac{1}{\omega_0}\arcsin\left(\frac{\omega_0}{\lambda}\left(\alpha - \frac{\beta K^2}{\omega_0^4}\right)\right) \quad (29)$$

and

$$T > \frac{1}{\omega_0}\left(\pi - \arcsin\left(\frac{\omega_0}{\lambda}\left(\alpha - \frac{\beta K^2}{\omega_0^4}\right)\right)\right). \quad (30)$$

On the other hand, to find the frequency of the limit cycle, we let $\psi = \omega_0 t + \varphi$ denotes the argument of the {\rm cos}ine in equation 12. Then the frequency of the periodic solution is

$$\omega = \frac{d\psi}{dt} = \omega_0 + \frac{d\varphi}{dt}. \quad (31)$$

Using equation 26 yields

$$\omega = \omega_0 - \frac{3K^2}{2\omega_0^3}\delta(1+2(a\Omega)^2S^2) - \frac{3(a\Omega)^2}{2\omega_0^3}(\gamma+\delta)CSK + \frac{3K}{2\omega_0^5}\gamma^2(a\Omega)^2C^2 - \frac{\lambda}{2\omega_0}\cos\omega_0 T + \frac{3}{8\omega_0}(\gamma^2(a\Omega)^2C^2 - \delta(1+2(a\Omega)^2S^2))R^2. \quad (32)$$

Equation 32 gives a relationship between the frequency of the limit cycle ω, the excitation frequency Ω and the delay period T. A condition for the existence of the limit cycle is guaranteed when the frequency ω is positive. This means that the following conditions, denoted by (C2), obtained from equation 32, must be satisfied

$$T > \frac{-i}{\omega_0}\ln\left(\frac{\mathrm{EF} - \mathrm{i}\sqrt{\mathrm{F}^2(\mathrm{G}^2+\mathrm{F}^2-\mathrm{E}^2)}}{\mathrm{F}(-\mathrm{G}+\mathrm{iF})}\right) \quad (33)$$

and

$$T < \frac{-i}{\omega_0}\ln\left(\frac{\mathrm{EF} + \mathrm{i}\sqrt{\mathrm{F}^2(\mathrm{G}^2+\mathrm{F}^2-\mathrm{E}^2)}}{\mathrm{F}(-\mathrm{G}+\mathrm{iF})}\right), \quad (34)$$

where E, F and G are given in the Appendix and $i = \sqrt{-1}$. The relation

$$\omega_0^2 = 1 + \frac{(a\Omega)^2}{2}(S^2 - 2\gamma C^2)$$

provides the third condition (C3):

$$\Omega < \frac{1}{a}\sqrt{\frac{2}{2\gamma C^2 - S^2}}. \quad (35)$$

The conditions (C1), (C2) and (C3), given by equations 29 and 30, equations 33 and 34, and equation 35, respectively, are illustrated in Figure 1. These three combined conditions delimit in the parameter plane (T, Ω), by solid lines, the existence zone of limit cycle. As it can be seen in Figure 1, the suppression region of self-excited oscillations (region B) decreases by increasing the incline angle θ. Figure 1 also suggests that the incline angle of the fast excitation can control the domain of existence and non existence of limit cycle. Validation of these analytical results by numerical integrations was done in Sah and Belhaq (2008), Belhaq and Sah (2008a) and Belhaq and Sah (2008b).

3. Limiting Cases

In the case of a horizontal excitation, $\theta = 0$ $(C = 1, S = 0)$, the autonomous equation governing the slow dynamic of motion takes the form Belhaq and Sah (2008b)

$$\ddot{z} - (\alpha - \beta z^2)\dot{z} + \omega_0^2 z + ((a\Omega)^2\gamma^2 - \delta)z^3 = \lambda z(t-T), \quad (36)$$

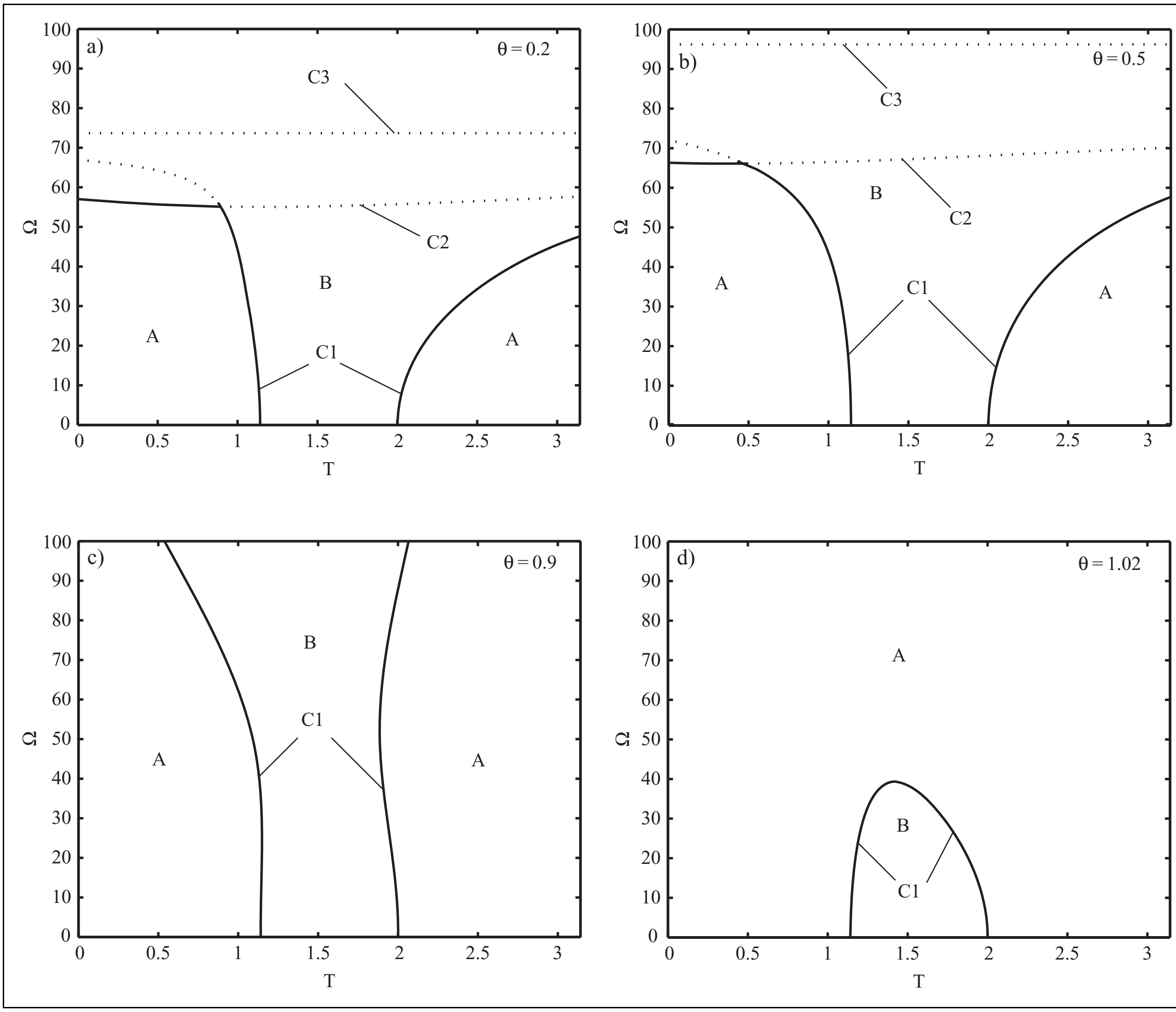

Figure 1. Suppression domain of limit cycle: $a = 0.02$, $\alpha = \beta = \gamma = 0.5$, $\delta = 1/6$, $\lambda = 0.55$ and for different values of the incline θ. Region A: limit cycle exists. Region B: absence of limit cycle.

where $\dot{z} = dz/dt$ and $\omega_0^2 = (1 - (a\Omega)^2\gamma)$. Setting $\theta = 0$ in equations 25 and 26 we obtain the slow flow

$$\dot{R} = \left(\frac{\alpha}{2} - \frac{\lambda}{2\omega_0}\sin\,\omega_0 T\right)R - \frac{\beta}{8}R^3, \tag{37}$$

$$\dot{\varphi} = -\frac{\lambda}{2\omega_0}\cos\,\omega_0 T + \frac{3}{8\omega_0}(\gamma^2(a\Omega)^2 - \delta)R^2 \tag{38}$$

and the conditions (C1), equation 29 and 30, read

$$T < \frac{1}{\sqrt{1 - (a\Omega)^2\gamma}}\,\arcsin\left(\frac{\alpha}{\lambda}\sqrt{1 - (a\Omega)^2\gamma}\right) \tag{39}$$

and

$$T > \frac{1}{\sqrt{1 - (a\Omega)^2\gamma}}\left(\pi - \arcsin\left(\frac{\alpha}{\lambda}\sqrt{1 - (a\Omega)^2\gamma}\right)\right). \tag{40}$$

Equation 32 of the frequency of the periodic solution reduces to

$$\omega = \omega_0 - \frac{\lambda}{2\omega_0}\cos\,\omega_0 T + \frac{3}{8\omega_0}(\gamma^2(a\Omega)^2 - \delta)R^2 \tag{41}$$

and the conditions (C2), equations 33 and 34, take the form

$$T > \frac{-i}{\omega_0}\ln\left(\frac{EF - i\sqrt{F^2(G^2 + F^2 - E^2)}}{F(-G + iF)}\right) \tag{42}$$

and

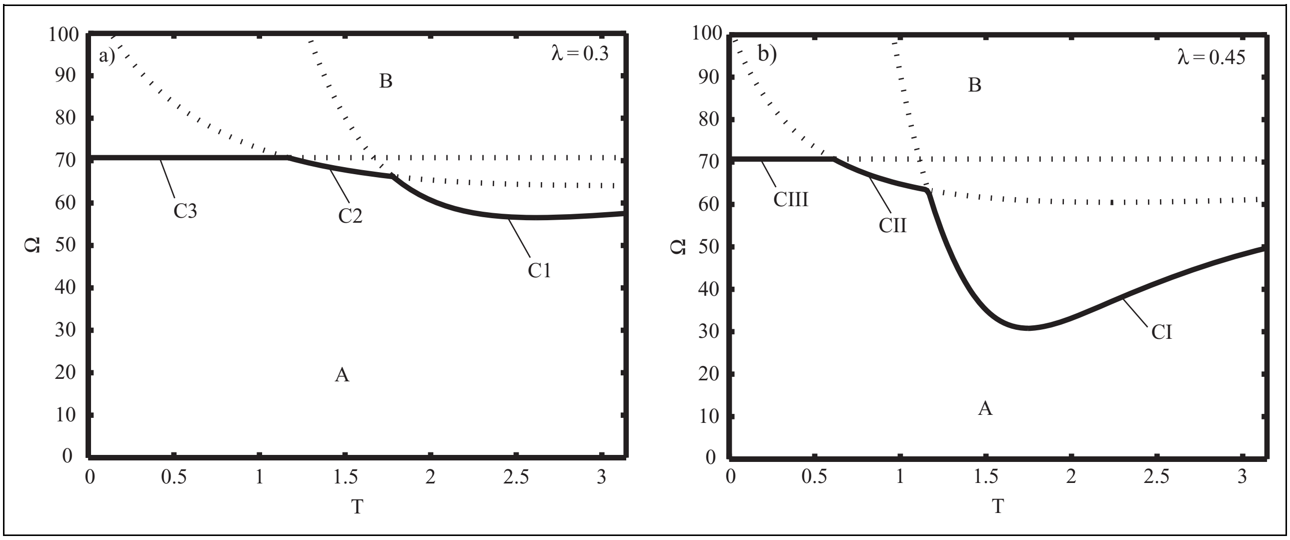

Figure 2. Suppression region of limit cycle in the case of horizontal excitation: $a = 0.02$, $\alpha = \beta = \gamma = 0.5$, $\delta = 1/6$: (a) $\lambda = 0.3$ and (b) $\lambda = 0.45$. Region A: limit cycle exists. Region B: absence of limit cycle.

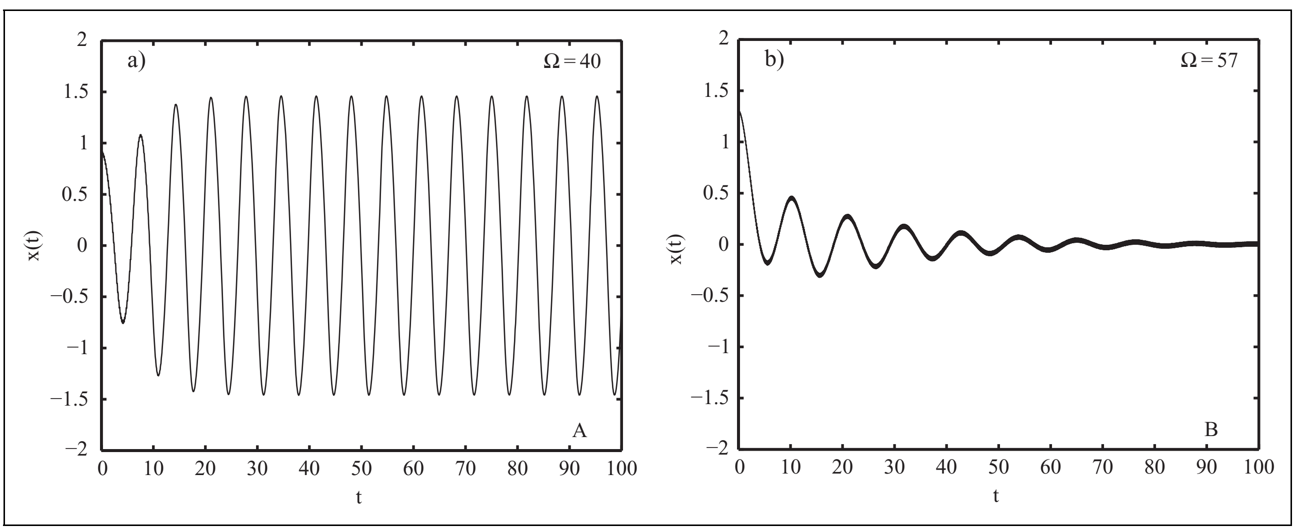

Figure 3. Time histories of the full motion $x(t)$ in the case of horizontal excitation with parameter values as for Figure 2(a) and $T = 2.5$. Here (a) and (b) correspond to regions A and B in Figure 2(a), respectively.

$$T < \frac{-i}{\omega_0} \ln\left(\frac{EF + i\sqrt{F^2(G^2 + F^2 - E^2)}}{F(-G + iF)}\right), \tag{43}$$

where

$$E = \omega_0^2 + \frac{3\alpha}{2\beta}(\gamma(1 - \omega_0^2) - \delta), F = \frac{3\lambda}{2\beta\omega_0}(\gamma(\omega_0^2 - 1) + \delta),$$
$$G = -\frac{\lambda}{2} \quad \text{and} \quad i = \sqrt{-1}.$$

Finally, the condition (C3), equation 35, reduces to

$$\Omega < \frac{1}{a\sqrt{\gamma}}. \tag{44}$$

Figure 2 shows the three combined conditions (C1), (C2), and (C3) delimiting the suppression region of limit cycle in the parameter plane (T, Ω) for two different values of λ. From this figure, we note that the range of the time delay that has a stabilizing effect increases with the frequency of the fast excitation. It can be seen in Figure 2(b) that for large values of the delay amplitude λ, self-oscillations can be suppressed for moderate values of Ω near $T = \pi/2$. Time histories of the full motion obtained numerically (Shampine and Thompson, 2000) are shown in Figure 3 for values of Ω picked from Figure 2(a).

In the case of a vertical excitation (C=0, S=1), the equation governing the slow dynamic of motion is given by (Belhaq and Sah, 2008a,b)

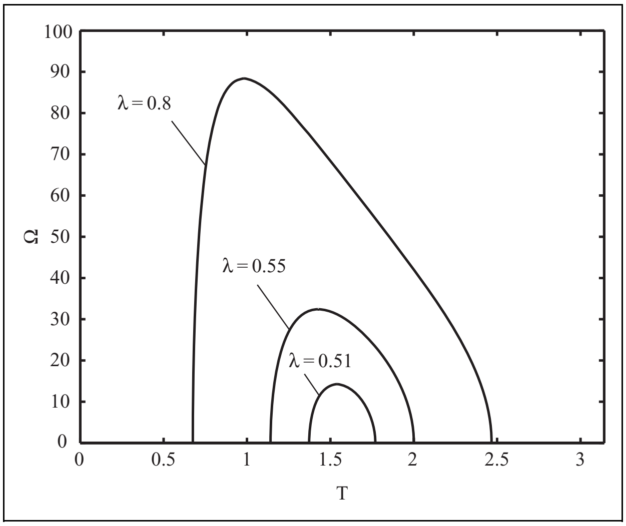

Figure 4. Suppression region of limit cycle (below the curves) in the case of vertical excitation: $a = 0.02$, $\alpha = \beta = 0.5$.

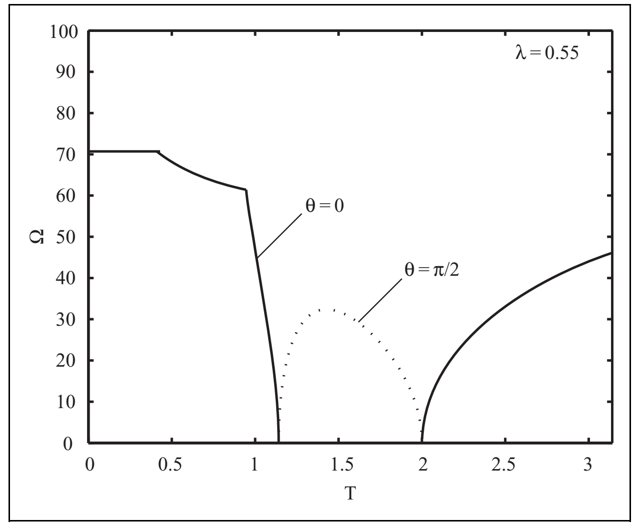

Figure 5. Suppression domain of the limit cycle: $a = 0.02$, $\alpha = \beta = \gamma = 0.5$. Horizontal case (solid lines): no limit cycle between solid lines. Vertical case (dotted line): no limit cycle below dotted line.

$$\ddot{z} - (\alpha - \beta z^2)\dot{z} + \omega_0^2 z - \delta(1 + 2(a\Omega)^2)z^3 = \lambda z(t - T), \tag{45}$$

where $\omega_0^2 = 1 + (a\Omega)^2/2$. The slow flow in the vertical case is

$$\dot{R} = (\frac{\alpha}{2} - \frac{\lambda}{2\omega_0} \sin \omega_0 T)R - \frac{\beta}{8} R^3, \tag{46}$$

$$\dot{\varphi} = -\frac{\lambda}{2\omega_0} \cos \omega_0 T - \frac{3\delta}{8\omega_0} (2(a\Omega)^2 + 1)R^2. \tag{47}$$

The two conditions on the time delay T ensuring that the amplitude

$$R = \sqrt{\frac{8}{\beta}\left(\frac{\alpha}{2} - \frac{\lambda}{2\omega_0} \sin \omega_0 T\right)}$$

of the limit cycle is real are

$$T < \frac{1}{\sqrt{1 + \frac{(a\Omega)^2}{2}}} \arcsin\left(\frac{\alpha}{\lambda}\sqrt{1 + \frac{(a\Omega)^2}{2}}\right), \tag{48}$$

$$T > \frac{1}{\sqrt{1 + \frac{(a\Omega)^2}{2}}}\left(\pi - \arcsin\left(\frac{\alpha}{\lambda}\sqrt{1 + \frac{(a\Omega)^2}{2}}\right)\right). \tag{49}$$

The frequency of the limit cycle given by

$$\omega = \omega_0 - \frac{\lambda}{2\omega_0} \cos \omega_0 T - \frac{3\delta}{8\omega_0}(2(a\Omega)^2 + 1)R^2. \tag{50}$$

The conditions in equations 48 and 49 are valid when the inequality

$$\frac{\alpha}{\lambda}\sqrt{1 + \frac{(a\Omega)^2}{2}} \le 1$$

is held. In Figure 4 we plot these conditions for different values of λ. These curves suggest that for a fixed value of the delay amplitude, there exists a threshold value of Ω for which a self-excited oscillation appears. This is given by

$$\Omega = \frac{1}{a}\sqrt{2\left(\frac{\lambda^2}{\alpha^2} - 1\right)}. \tag{51}$$

It can be seen from Figure 4 that as the delay amplitude increases, the stabilizing effect increases as well.

Finally, Figure 5 shows the results obtained for the horizontal ($\theta = 0$, solid lines) and the vertical ($\theta = \pi/2$, dotted line) excitation cases. In the horizontal excitation case, the zone where the limit cycle is absent is located between the two solid lines, whereas this zone is located below the dotted line in the vertical case.

4. Conclusion

We have studied the tilting effect of a FHE on self-excited vibrations in a delayed van der Pol oscillator. We have used averaging techniques to derive the slow flow equations. This slow flow has been analyzed and the effect of the tilted angle of the excitation has been discussed. It has been shown that when the incline angle increases from the horizontal, the suppression domain of self-excited vibrations decreases. It has also been shown that in the case of horizontal excitation, the range of the time delay that has stabilizing effect increases with the frequency of the fast excitation (Figure 2). The results suggest that in some practical situations in which the delay is imposed and the fast excitation is induced by an external vibrational environment, the tilt angle of the excitation can control the domain of existence and nonexistence of limit cycle depending on whether self-excited oscillations are desired or not.

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Appendix

Expressions of E, F and G in equations 33 and 34 are

$$E = \omega_0^2 - \frac{3K^2}{2\omega_0^2}\delta(1+2(a\Omega)^2S^2) - \frac{3(a\Omega)^2}{2\omega_0^2}(\gamma+\delta)CSK + \frac{3K}{2\omega_0^4}\gamma^2(a\Omega)^2C^2 + \frac{3}{2\beta}\left(\alpha - \frac{\beta K^2}{\omega_0^4}\right)(\gamma^2(a\Omega)^2C^2 - \delta(1+2(a\Omega)^2S^2)), \tag{52}$$

$$F = -\frac{3\lambda}{2\beta}(\gamma^2(a\Omega)^2C^2 - \delta(1+2(a\Omega)^2S^2)), \tag{53}$$

$$G = -\frac{\lambda}{2}. \tag{54}$$

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